# Learning and the emergence of nonlinearity in financial markets

Ian Dew-Becker, Stefano Giglio, and Pooya Molavi*

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**Abstract**

Financial markets (and more generally the real economy) display a wide range of important nonlinearities. This paper focuses on stock returns, which are skewed left – generating crashes – and whose volatility moves over time, is itself skewed, is strongly related to the level of prices, and displays long memory. This paper shows that such behavior is almost inevitable when prices are formed by investors acquiring information about the true, but latent, value of stocks. It studies a general model of filtering in which agents receive signals about the fundamental value of the stock market and dynamically update their beliefs (potentially with biases). When those beliefs are non-normal and investors believe crashes can happen, prices generically display the range of nonlinearities observed in the data. While the model does not explain where crashes come from, it shows that investors believing that prices can crash is sufficient to generate the rich higher-order dynamics observed empirically. In a simple calibration with iid shocks to fundamentals, the model fits well quantitatively, and regression-based tests support the model's mechanism.

# 1 Introduction

### Background and contribution

*Dew-Becker: Federal Reserve Bank of Chicago; Giglio: Yale University; Molavi: Northwestern University. The views in this paper are those of the authors and do not represent those of the Federal Reserve System or Board of Governors. We appreciate helpful comments and discussions from Rui Da, Jesse Davis, Sergei Glebkin, Jiantao Huang, Paymon Khorrami, Lars Lochstoer, Jean-Paul Renne, Larry Schmidt, and seminar participants at the Bank of Canada, Duke, Stanford, Indiana, Yale, the San Francisco and Chicago Feds, the IMF, and the EFA, Transatlantic Theory, CEPR Beliefs and the Macroeconomy, INSEAD, FIRS, HKU Macro, Saieh Fellows, Carey Finance, and NBER Asset Pricing conferences.

Stock market returns are far from normally distributed. The most salient deviation is that the market sometimes crashes, skewing the distribution of returns left. Crashes and negative skewness more generally have been the subject of huge amounts of research trying to understand both their causes and consequences. For example, do stock market crashes cause declines in GDP or does the causation run the opposite direction? Does crash risk explain the equity premium?[1]

But stock market returns are much more complicated than simply being independent draws from a negatively skewed distribution. Their volatility fluctuates over time, and those fluctuations have also been the subject of large literatures in both macroeconomics and finance.[2] Movements in volatility are themselves positively skewed and they have a strong negative correlation with market returns (about -80%) known as the **leverage effect**.[3] The strength of that relationship also changes over time and is related to the conditional skewness of market returns (Neuberger (2012)).[4] Finally, volatility has nonlinear dynamics: following large increases, instead of decaying geometrically, it tends to revert initially relatively quickly and then more slowly.[5]

The basic contribution of this paper is to show that not only is it not surprising that the stock market behaves in those ways, but that such behavior is nearly inevitable when stock prices are set by investors who are trying to learn about the true value of stocks.[6] While there is work that has studied different aspects of stock return nonlinearity individually, this paper is the first to propose a joint explanation. Its analysis cannot explain *why* there are crashes; instead what it shows is that in a world where crashes happen and investors are continually acquiring information, those crashes should happen in a consistent way: with volatility rising as prices fall and then nonlinearly returning to its mean, and the strength of

[1]On bubbles and crashes, among many others, see Abreu and Brunnermeier (2003) and Phillips, Shi and Yu (2015). For their relationship with GDP, see Reinhart and Rogoff (2009) and Sufi and Taylor (2022). On crash risk and the equity premium, see Rietz (1988) and Barro (2006).

[2]E.g. Bloom (2009) studying shocks to the VIX and the subsequent literature that follows. More recently, see Caldara, Fuentes-Albero, Gilchrist and Zakrajšek (2016) and Ludvigson, Ma and Ng (2021).

[3]See Merton (1980), French, Schwert and Stambaugh (1987), and Cont (2001). The term is generally viewed as a misnomer – while financial leverage can qualitatively generate countercyclical volatility, the effect would be smaller than what is observed empirically by an order of magnitude.

[4]For recent work on skewness and the macroeconomy, see Salgado, Guvenen and Bloom (2020), Gormsen and Jensen (2023), Iseringhausen, Petrella and Theodoridis (2023), Dew-Becker (2024), and Menkhoff (2025).

[5]E.g. Mandelbrot (1963), Granger (1980), and Mandelbrot, Fisher and Calvet (1997). Cont (2001) gives a thorough and still relevant review of the facts for volatility dynamics and other nonlinearities in financial markets.

[6]For past work on belief dynamics and asset prices, among many others, see David (1997), Veronesi (1999), Weitzman (2007), David and Veronesi (2013), Collin-Dufresne, Johannes and Lochstoer (2016), Gennaioli, Shleifer and Vishny (2015), Johannes, Lochstoer and Mou (2016), Crump, Eusepi and Moench (2018), Kozlowski, Veldkamp and Venkateswaran (2018), Farmer, Nakamura and Steinsson (2024), Wachter and Zhu (2023), and Orlik and Veldkamp (2024).

that relationship being related to the conditional skewness of returns. Information processing naturally and almost inevitably creates the pattern of nonlinearities observed in asset prices.

**Methods**

The paper's theoretical structure is built around the premise that agents want to know the discounted value of a security's cash flows. It uses a very general setup: the net present value (NPV) follows some arbitrary process, and agents continuously receive signals about it. Since the NPV process is essentially unconstrained, the analysis nests a wide range of specifications that have been studied in the literature. A strength of the paper is in being able to collect a wide range of features of learning that have been studied in specific settings in a single unified and general framework.

Asset prices in the model are the solution to a filtering problem. Agents observe signals about the true value of stocks, which they use to infer a posterior distribution on each date. The level of the stock market is their posterior mean.

The paper's core theoretical tool is a novel result showing that belief dynamics have a simple recursive structure: the sensitivity of the posterior mean to signals is equal to the posterior variance multiplied by signal precision, and the sensitivity of the posterior variance is equal to the posterior third moment times signal precision. The result also yields an expression for the mean reversion in the posterior variance. In fact those claims are not restricted to an asset pricing setting; they apply to dynamics of beliefs more generally. Finally, they also do not require full rationality – agents' beliefs about crash risk or the persistence of fundamentals, for example, could be misspecified.

**Results**

The first important feature of the theoretical results is that they immediately imply there is a tight relationship between investors' uncertainty about fundamentals and return volatility – high uncertainty in the model creates high return volatility, since the level of uncertainty determines how strongly agents respond to signals they observe.

That then helps understand the leverage effect – the increase in volatility as prices fall. There is a necessary and sufficient condition for the leverage effect to appear in the model: agents must have negatively skewed beliefs about fundamentals. When agents' subjective distribution over the true value for fundamentals is negatively skewed, a negative signal – which drives their mean, and hence prices, down – also raises uncertainty, since negative skewness means that the left-hand side of their distribution is wider than the right. So when investors have negatively skewed beliefs, negative news simultaneously reduces prices and raises future volatility. And that happens generically, including in models in which the fundamentals process has constant volatility.

It is again important to note that the paper does not take a stand on why investors have negatively skewed beliefs. Naturally that skewness may come from the facts that crashes do happen and that other features of the economy also display negative skewness.[7] But behavioral biases or a type of ambiguity aversion could also play a role.

Unsurprisingly, the strength of the leverage effect in the model is related to the magnitude of skewness in beliefs. In a simple empirical analysis looking at the US stock market and natural gas futures (with the latter selected for having strongly *positive* skewness in contrast to the stock market), the leverage effect coefficient lines up strikingly well with the model's quantitative prediction.

The paper next shows that nonlinear filtering also generically yields long memory in volatility, i.e. nonlinear decay following shocks. When uncertainty is high, agents put high weight on the signals they receive, causing them to learn and reduce uncertainty quickly. But as uncertainty falls, they also naturally give each additional signal less weight, causing their learning to slow. The result is that following upward jumps, the rate of mean reversion in volatility is high initially and then slows, consistent with the data. The paper shows that this behavior is a generic feature of models with learning, but it has not previously been highlighted as such.

After developing a few more theoretical results, the paper moves on to a quantitative analysis that examines the model's predictions from two perspectives. First, since the theoretical results are primarily qualitative statements about limits, we examine a simple calibration to quantify the effect sizes. The calibration is set up so that the latent value of stocks follows an iid disaster process with jump sizes distributed as in the estimates of Barro and Jin (2011) for global consumption disasters, while the signal precision is taken as a free parameter to match the data. Importantly, fundamentals follow a random walk in the calibration so that without learning there would be no nonlinearities in returns other than skewness.

The calibration provides a good fit to the first four moments of returns and volatility. It also fits the autocorrelations in volatility and the magnitude of the leverage effect. The calibration shows that the addition of an extremely simple learning process to a standard disaster setup generates quantitatively significant dynamics that match the data well.

It is also important to note that many of the statistics that the calibration matches are *unconditional*. The nonlinear dynamics appear all the time, not just in crashes. The paper's

[7]For empirical work, see Sichel (1993), McKay and Reis (2008), Morley and Piger (2012), Berger, Dew-Becker and Giglio (2020), and Dupraz, Nakamura and Steinsson (2021). Theoretical work includes Ilut et al. (2018), Straub and Ulbricht (2019), Kozeniauskas et al. (2018), Gilchrist and Williams (2000), Kocherlakota (2000), Hansen and Prescott (2005), Bianchi (2011), and Bianchi et al. (2017).

observation is that when investors understand that crashes *can* happen, that influences how they process information and thus affects the dynamics of prices even in samples in which no disasters occur.

The second part of the quantitative analysis derives nonparametric predictions from the model. First, as discussed above, the model has predictions for the relationship between volatility, its own lag, returns, and conditional skewness that we test and find hold well in the data. Second, it is possible to estimate the precision of agents' signals and their implied uncertainty about the level of fundamentals, both without knowing the underlying model. In US stock market data, the data implies investors' conditional distribution for fundamentals has on average a standard deviation of 10.4–16.5 percent. In a survey administered by Yale University since the 1980's, cross-sectional disagreement about the fundamental value of the stock market has a standard deviation of 17.0 percent, which provides some independent support for the model-based estimate (subject to the usual caveat that disagreement and uncertainty are theoretically distinct).

**Implications**

The paper's core claim is that understanding the wide range of nonlinearities observed in stock market returns does not require a wide range of models. The behavior is consistent with a simple setup in which the true value of stocks drifts up over time and faces occasional large negative shocks, but investors only have noisy signals about that true value. While past work on learning has highlighted some of these mechanisms, this paper brings them into a unified framework and gives necessary and sufficient conditions for them to arise.

The tools and analysis developed here are applicable much more generally than just in the aggregate stock market. At a deep level they are really just about how beliefs evolve. The paper's methods are therefore relevant much more broadly in economics because they yield predictions for the evolution of beliefs in generic non-Gaussian settings (e.g. inflation and interest rates), even in the presence of potentially severe behavioral biases.

**Additional related work**

This paper provides unified results on the relationship between signals and beliefs. That makes it related to work on how uncertainty and responsiveness to news vary endogenously over time, such as van Nieuwerburgh and Veldkamp (2010), Ludvigson, Ma and Ng (2021), Ilut and Saijo (2021), Altig, Barrero, Bloom, Davis, Meyer and Parker (2022), and Bachmann, Carstensen, Lautenbacher, Menkhoff and Schneider (2024).[8] There is a large literature

[8]Additionally, there is a large related macro literature on endogenous uncertainty and volatility. For recent work, see Ordonez (2013), Fajgelbaum, Schaal and Taschereau-Dumouchel (2017), Straub and Ulbricht (2019), Ascari, Fasani, Grazzini and Rossi (2023), Straub and Ulbricht (2024), and Dew-Becker and Vedolin (2025).

on filtering and expectations formation in economics including, with varying degrees of rationality, Eusepi and Preston (2011), Nimark (2014), and Bianchi, Ilut and Saijo (2025).[9]

More directly, the analysis builds on models of asset pricing under learning including Veronesi (1999), David and Veronesi (2013), and Kozlowski, Veldkamp and Venkateswaran (2018). The first two papers study learning about states, while the last is about learning about time-invariant parameters, but both types of learning are accommodated within this paper's setup.[10]

Beyond the usual strict Bayesian setup, section 5.1 shows that the paper's analysis is also consistent with a number of deviations from the standard full information rational expectations setup that have been analyzed in the literature, such as misspecified or imperfect priors (e.g. Farmer, Nakamura and Steinsson (2024)) or errors in incorporating new data (e.g. Bordalo, Gennaioli, Porta and Shleifer (2019)).

The quantitative example the paper studies is closely related to work on rare disasters (e.g. Rietz (1988) and Barro (2006)). Even though the probability of a disaster (here just a jump in the fundamental value of stocks) is constant, at any given time agents are unsure whether a disaster has occurred, so their subjective distribution over future returns varies over time *as though* the probability of a disaster is time-varying, as in Gabaix (2012) and Wachter (2013) (see also Ghaderi, Kilic and Seo (2022)).[11]

**Outline**

The remainder of the paper is organized as follows. Section 2 presents empirical characteristics of returns that motivate the analysis. Section 3 describes the model structure and gives the main theoretical result. Section 4 then examines the theoretical predictions and section 5 studies some extensions and robustness to certain assumptions. Last, sections 6 and 7 take the model to the data, studying both a calibration and nonparametric tests of the theory, and section 8 concludes.

# 2 Motivating facts

We begin with a set of motivating facts.[12] All moments for realized returns are for the CRSP total market return in excess of the risk-free rate (from Kenneth French). Daily volatility is

[9]See also Adam and Marcet (2011) and Adam, Marcet and Nicolini (2016)

[10]See also Ai (2010), Bansal and Shaliastovich (2010), Abel, Eberly and Panageas (2013), Collin-Dufresne, Johannes and Lochstoer (2016), Ehling et al. (2018), and Andrei, Hasler and Jeanneret (2019), among many others.

[11]See also Wachter and Zhu (2023) for a model of learning with rare disasters and Maenhout, Xing and Balter (2025) for related work with ambiguity aversion. Baker, Bloom, Davis and Sammon (2025) provide evidence on what the actual events are that cause jumps in stock prices.

[12]Cont (2001) notes that similar behavior is observed across many different financial markets.

obtained by forecasting realized volatility using the VIX, so as to remove the time variation in risk premium.[13]

Table 1: Stock market return and volatility moments

| | Stock market | —— Volatility ——- | |
|---|---|---|---|
| Moment | Daily return | Level | Daily change |
| Std. dev. | 1.14 | 6.73 | 1.43 |
| Skewness | -0.06 | 2.16 | 1.37 |
| Kurtosis | 19.54 | 11.38 | 30.36 |
| Corr. w/ $R_t$ | | | -0.78 |

**Note:** The table reports empirical moments of stock market returns and volatility (level and daily changes). Volatility is the fitted value of a projection of realized volatility onto the VIX index and is reported in annualized standard deviation units.

Table 1 reports the variance, skewness, and excess kurtosis of daily market returns, their conditional volatility, and the daily change in that conditional volatility. Daily market returns are slightly negatively skewed, while volatility is highly positively skewed in both levels and changes. All three series have severe excess kurtosis.

The top-left panel of figure 1 plots the skewness of realized log returns over holding periods of 1 to 252 trading days. Skewness becomes significantly more negative as the horizon increases. It is -0.46 at the daily level -1.0 at a horizon of a year.

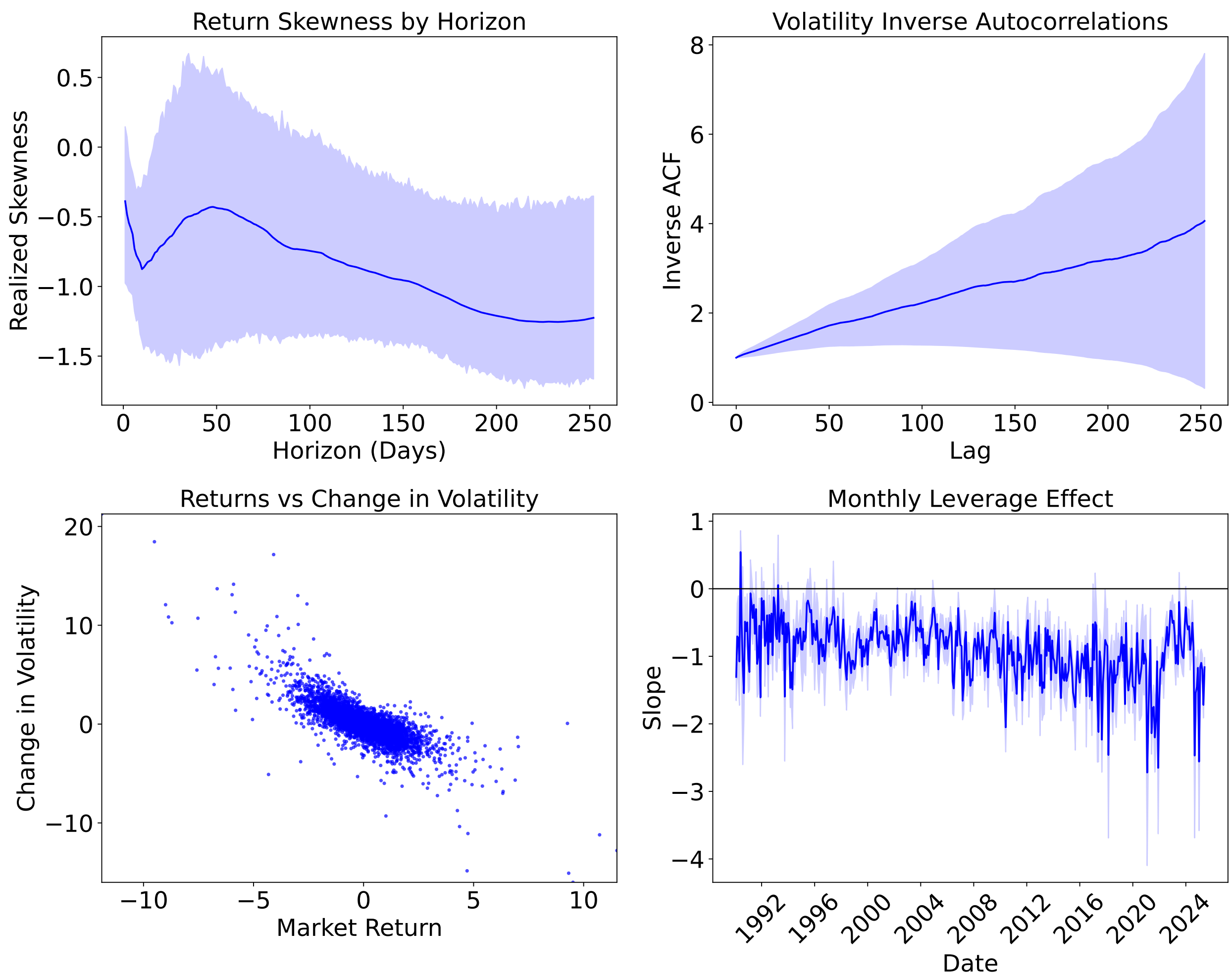
The bottom-left panel of figure 1 is a scatter plot of daily market returns against the daily change in the volatility, showing the strong negative correlation referred to as the leverage effect. The correlation coefficient in the sample is -0.78. The bottom-right panel plots estimates of the regression coefficient in every month between 1990 and 2025, showing that the negative relationship is not isolated to particular episodes – it holds in every month in the sample except for two, and also has been generally trending downward.

Finally, the top-right panel of figure 1 plots the inverse autocorrelations of volatility, $1/corr\left(vol_t, vol_{t-j}\right)$. The reason to plot the inverse autocorrelations is to help visualize the deviation from the exponential decay that would be expected if volatility followed an ARMA process. The fact that $corr\left(vol_t, vol_{t-j}\right)^{-1}$ grows approximately linearly is consistent with polynomial decay in the autocorrelations, as in fractionally integrated models like Ding,

[13]The regression forecasts is conditional just on what goes into the regression, so errors will arise if the conditioning set is too small. We do not find that typical cyclical variables provide any additional forecasting power beyond the VIX.

To be more specific, the conditional volatility of returns is the projection of realized return volatility onto current option-implied volatility (the VIX index). Conditional skewness is constructed using projections of realized skewness calculated as in Neuberger (2012).

Figure 1: Motivating evidence




**Note:** Top left: realized skewness of log returns computed over different holding periods. Top right: inverse of autocorrelations of volatility at different lags (in days). Bottom left: scatterplot of daily changes in volatility against daily market returns. Bottom right: monthly series of the slope of a regression (using daily data within each month) of change in volatility onto returns. All figures use daily data from 1990. Shaded areas are 95% confidence intervals.

Granger and Engle (1993) and Bollerslev and Mikkelsen (1996).[14]

# 3 Model setup and solution

## 3.1 Model setup

### 3.1.1 Dynamics of fundamentals

Stocks pay some cash-flow $D_t$ and there is a stochastic discount factor $M_t$ such that, conditional on an information set $\mathcal{I}_t$, prices satisfy

$$P_t(\mathcal{I}_t) = \mathbb{E}\left[\int_{s=0}^{\infty} \frac{D_{t+s}M_{t+s}}{M_t} ds \mid \mathcal{I}_t\right] \tag{1}$$

where $\mathbb{E}$ is the expectation operator and the notation $P_t(\mathcal{I}_t)$ emphasizes the dependence of prices on the information set. This specification nests a wide range of models (a Lucas tree economy being the simplest example) – the stochastic discount factor encodes risk aversion and any other drivers of state prices and can, under certain assumptions, also represent distortions in beliefs. The simplest version of the analysis holds when $D$ and $M$ are exogenous to agents' information and we assume that for the baseline results. That assumption rules out, for example, models in which marginal utility depends on prices, such as the CAPM (see section 5.3).

At any given time, the full set of information that an agent could possibly know is represented by some abstract object $\theta_t$ (which might be a scalar, a vector, a function, or something more exotic). The **fundamental value** of the asset is its price conditional on complete knowledge of $\theta_t$,

$$X_t \equiv P_t(\theta_t) \tag{2}$$

An extreme case is perfect foresight, in which $\theta_t$ contains complete knowledge of all values of cash flows in the future, but $\theta_t$ can also be coarser.

The model is driven by the dynamics of the state variable $\theta_t$ and the function $P_t$ mapping from information to asset prices. Assumptions 3–5 in the appendix give the required technical restrictions, which are simply those necessary for filtering to be possible. Essentially, the first and second moments of $X_t$ (and its Fourier transform) need to exist and not be too pathological. The jump diffusions and continuous-time ARMA processes, both in scalar and vector forms, that are typically studied in economics will be acceptable here.

[14] Though note that fractional integration is an asymptotic concept and there are infinitely many ARMA models that can match any finite set of autocorrelations.

Finally, note that while $X$ itself is a scalar, it is a function of $\theta_t$, meaning its dynamics can be very rich and can accommodate, for example, many forms of nonstationarity.

#### 3.1.2 Information flows

Agents' information can be represented by a sufficient statistic $Y$ whose history is denoted by $Y^t$. If the payoff-relevant information in $Y^t$ is a subset of that in $\theta_t$ (i.e. $P_t(\{Y^t, \theta_t\}) = P_t(\theta_t)$), then by the law of iterated expectations,

$$P_t\left(Y^t\right) = \mathbb{E}\left[X_t \mid Y^t\right] \tag{3}$$

Equation (3) is a standard filtering problem. Prices are a simple expectation here because $X$ itself already includes risk adjustments represented in state prices via the $M$ process.

The one final wrinkle is that because aggregate stock prices display trend growth, they are typically modeled in logs. If $X$ is an arithmetic process (which it might be in the case of inflation or interest rates), we could directly apply (3). To analyze stock prices, which are typically modeled as a geometric process, we restate the filtering problem in logs.

**Assumption 1** *In the theoretical analysis, the log price, $p_t$, is*

$$p_t \equiv \mathbb{E}\left[x_t \mid Y^t\right] \tag{4}$$

*where $x_t$ is the log NPV process,*

$$x_t \equiv \mathbb{E}\left[\log \int_{s=0}^{\infty} \frac{D_{t+s} M_{t+s}}{M_t} ds \mid \theta_t\right] \tag{5}$$

That act of passing the log through the expectation is the single approximation step in the theoretical analysis. Transformations like this are not uncommon – analyses of macro-finance models very often rely on the Campbell–Shiller approximation, for example. Appendix A.3.4 reports analogs to the main results below without applying (4) and shows that they are similar, and the quantitative analysis in section 6 does not use this approximation.

**Assumption 2** *Agents' information can be summarized by a sufficient statistic $Y$ satisfying*

$$dY_t = x_t dt + \sigma_{Y,t} dW_t \tag{6}$$

*where $\sigma_{Y,t}$ follows some exogenous process (subject to assumption 5 in appendix A.1.1).*

The setup in assumption 2 can accommodate a number of different underlying information structures. First, agents might have access to a number (as small as 1) of independent Gaussian sources of information (e.g. $Y_{1,t}$, $Y_{2,t}$, etc.) all of the form in (6), which then can always be summarized by a single linear combination which will itself also satisfy (6). Another possibility is that agents have many individual sources of information, none of which are Gaussian and all of which are asymptotically only weakly informative. Such signals can be combined via a martingale central limit theorem into a single composite that converges to the form (6). $Y$ might also be taken as the outcome of an information processing (or rational inattention) type of problem in which it represents the information that agents are ultimately able to use for decisions. Appendix A.3.1 also shows that (6) can obtain under some scenarios where agents observe only cash flows.

That said, our preferred interpretation is that $Y$ is a composite representing the combination of the broad range of sources investors have access to, especially given the well known difficulty of connecting most movements in stock prices to any specific signal (e.g. Roll (1988)). The reasonableness of assumption 2 under that interpretation should be judged based on its observable predictions for the behavior of endogenous outcomes like prices.

Obviously assumption 2 is far from the only possible information structure. Agents could receive signals about nonlinear functions of $x_t$, such as its moments, or about $\theta_t$, which might contain relevant information about the future path of $x$. Additionally, they might draw inferences about $\theta_t$ from realized cash-flows.[15] The $Y$ process is meant to capture all the information agents receive in a single factor. And given that $x$ represents how agents would value stocks if they had complete information, it makes a certain amount of sense to assume that it is what agents learn about. The assumption that information flows diffusively also matters for the analysis, but it is not completely restrictive – see section 5.2. The analysis is extremely general in the dynamics for fundamentals, represented by $x$, but pays for that generality with this restriction on the information structure.

Up to equation (6), there is little loss of generality in the analysis other than the restriction that $M$ and $D$ are exogenous to information. The choice of the information structure is where the model is significantly restricted. Section A.3.5 examines how to get to equation (4) in a Lucas tree type economy with the information structure in (6).

Finally, while the setup is motivated by an asset pricing problem, it is much more general. $x$ is just some latent object of interest – it could be trend inflation, for example. The following results are general statements about nonlinear filtering, not just asset price dynamics.

[15]Note that in the case of US stocks, cash-flows are strictly pre-determined. Dividends, for example, are announced well in advance of their payment.

### 3.2 Solution to the filtering problem

**Proposition 1** *Given (6) and restrictions on $x_t$ given in appendix A.1, the posterior mean ($\kappa_{1,t}$) and variance ($\kappa_{2,t}$) satisfy*

$$dp_t = d\kappa_{1,t} = \kappa_{2,t}\sigma_{Y,t}^{-2}\left(dY_t - \mathbb{E}_t\left[x_t\right]dt\right) + \mathbb{E}_t\left[dx_t\right] \tag{7}$$

$$\begin{aligned} d\,\mathrm{var}_t\left[x_t\right] &= d\kappa_{2,t} = \kappa_{3,t}\sigma_{Y,t}^{-2}\left(dY_t - \mathbb{E}_t\left[x_t\right]dt\right) - \kappa_{2,t}^2\sigma_{Y,t}^{-2}dt \\ &\quad + \mathbb{E}_t\left[d\left\langle x\right\rangle_t\right] + 2\,\mathrm{cov}_t\left(x_t, dx_t\right) \end{aligned} \tag{8}$$

*where $\kappa_{3,t}$ is the posterior third moment, $\langle x\rangle$ is the total quadratic variation process of $x$, and $\mathbb{E}_t[\cdot] \equiv \mathbb{E}[\cdot \mid Y^t]$.*

For the mean, $\kappa_{1,t}$, the first term says that the sensitivity to news is equal to current uncertainty ($\kappa_{2,t}$) multiplied by the precision of the signal, while the second term is simply the current expected drift. The intuition for the gain is simple: $\kappa_{2,t}/\sigma_{Y,t}^2 = \mathrm{cov}_t\left(x_t, dY_t\right)/\mathrm{var}_t\left(dY_t\right)$ is the coefficient from a hypothetical local regression of $x$ on $dY$.

The dynamics of the conditional variance are similar. The gain is now the *third* moment times the precision of the signal. That is again because $\kappa_{3,t}/\sigma_{Y,t}^2$ is a local regression coefficient. We discuss the drift term $\kappa_{2,t}^2\sigma_{Y,t}^{-2}dt$ further below. $\mathbb{E}_t\left[d\left\langle x\right\rangle_t\right]$ represents the expected accumulation of variance in $x$ (i.e. due to shocks), and $2\,\mathrm{cov}_t\left(x_t, dx_t\right)$ is the accumulation of uncertainty due to $x$ effectively "spreading out" over time.

**Remark 1** *Prices follow an Itô diffusion and satisfy*

$$\begin{aligned} dp_t &= \mu_{p,t}dt + \sigma_{p,t}d\tilde{W}_t \\ \textit{where } d\tilde{W}_t &\equiv \sigma_{Y,t}^{-1}\left(dY_t - \mathbb{E}_t\left[x_t\right]dt\right) \\ \mu_{p,t}dt &= \mathbb{E}_t\left[dx_t\right],\ \sigma_{p,t} = \kappa_{2,t}/\sigma_{Y,t} \end{aligned} \tag{9}$$

*and $\tilde{W}_t$ is a Brownian motion with respect to the agent's filtration.*

Prices follow a completely standard continuous diffusive process. What the model delivers is simply a very specific structure for the conditional volatility. All of the predictions for prices ultimately follow from the dynamics of volatility, which is itself a diffusion driven by the same Brownian motion $\tilde{W}$ (see corollary 2 below).

#### 3.2.1 General result

While proposition 1 is enough for the present paper, it suggests a broader result: the gain coefficients seem to satisfy a recursion. That recursion turns out to hold for the cumulants of agents' posteriors, which are the derivatives of the log characteristic function. The first three cumulants are equal to the first three central moments. Denote the $n$-th cumulant of the time-$t$ conditional distribution of $x_t$ by $\kappa_{n,t}$.

**Theorem 1** *Under the conditions of proposition 1, for all $n$ for which the $n+1$th cumulant exists*[16]

$$d\kappa_{n,t} = \kappa_{n+1,t}\sigma_{Y,t}^{-2}\left(dY_t - \mathbb{E}_t[x_t]dt\right) - \frac{1}{2\sigma_{Y,t}^2}\sum_{j=2}^{n}\binom{n}{j-1}\kappa_{j,t}\kappa_{n-j+2,t}dt + \sum_{j=1}^{n}\binom{n}{j}B_{n-j}\left(-\kappa_{1,t},\ldots,-\kappa_{n-j,t}\right)\mathbb{E}_t[d(x_t^j)] \quad (10)$$

*where $B_j$ denotes the $j$th complete exponential Bell polynomial.*

The result follows from an application of textbook results in Liptser and Shiryaev (2013) and Bain and Crisan (2009), but the statement in terms of cumulants – which is precisely what makes it useful for our purposes – is novel to this paper, as far as we can tell.[17] The recursion for the gain carries through to all the cumulants: the gain of the $n$th cumulant is the $(n+1)$th cumulant times the precision of the signal.

# 4 Predictions

This section examines the predictions of proposition 1 for the behavior of returns. Many of the results pertain to return volatility, which we define as the instantaneous volatility process for log prices[18]

$$vol_t \equiv \left(\lim_{\Delta t\downarrow 0}\mathbb{E}\left[\left(p_{t+\Delta t} - \mathbb{E}_t p_{t+\Delta t}\right)^2\right]/\Delta t\right)^{1/2} \quad (11)$$

$$= \kappa_{2,t}/\sigma_{Y,t} \quad (12)$$

[16]Since the cumulants are derivatives of a function, if $\kappa_{n+1,t}$ exists then all lower-order cumulants also exist. Note that the distribution of $x_t$ conditional on $Y^t$ is necessarily subgaussian, meaning that all moments and cumulants exist (Guo, Wu, Shamai and Verdú (2011)). So the restriction to $n$ such that the $n+1$th cumulant exists may possibly be satisfied for all $n$ for all processes, but we have not been able to verify that.

[17]Theorem 1 is closely related to results in Dytso, Poor and Shitz (2022), with two key differences. First, $x_t$ here is dynamic instead of constant. Second, theorem 1 enables the calculation of the evolution of the conditional cumulants from knowledge only of the priors. While there do not appear to be any other direct precedents to the results in their work and ours, we find it unlikely that nobody else did a similar calculation at some point.

[18]For stocks at high frequency, cash flows are predetermined, and in any case the variance of changes in cash flows for the aggregate US stock market at even the monthly frequency is insignificant compared to changes in prices. The historical variance of monthly returns is $2.85\times10^{-3}$, while the variance of dividend growth is over 600 times smaller – $4.46\times10^{-6}$. We therefore treat return volatility as equal to price volatility.

That is, $vol_t$ is simply the diffusive volatility of prices from the representation (9). Again, the conditional volatility of prices depends on agents' current posterior variance over fundamentals, $\kappa_{2,t}$. So, up to $\sigma_{Y,t}$, price volatility measures uncertainty.

Combining equations (7) and (8) from proposition 1 yields the following.

**Corollary 2** *$vol_t$ follows a diffusion satisfying*

$$
\begin{aligned}
d\left(vol_t\right) &= \underbrace{\sigma_{Y,t}^{-1}\frac{\kappa_{3,t}}{\kappa_{2,t}}\left(dp_t - \mathbb{E}_t\left[dp_t\right]\right)}_{ⓐ} - \underbrace{\sigma_{Y,t}^{-2} vol_t^2 dt}_{ⓑ} \\
&\quad + \sigma_{Y,t}^{-1}\left(\mathbb{E}_t\left[d\left\langle x\right\rangle_t\right] + 2\operatorname{cov}_t\left(x_t, dx_t\right)\right) - \sigma_{Y,t}^{-1} vol_t d\sigma_{Y,t} \qquad (13)
\end{aligned}
$$

The main predictions arise out of the terms ⓐ and ⓑ. ⓐ determines the joint behavior of returns and volatility, while ⓑ generates nonlinearity in the dynamics of volatility. The terms on the second line again involve the spreading out of $x_t$ itself along with the dynamics of $\sigma_{Y,t}$, both of which are exogenous, as opposed to coming from learning.[19]

A first point to note is that volatility is generically time-varying. It is only when the model is fully linear and Gaussian that the volatility of prices converges to a constant. If any of the higher-order cumulants is nonzero, that effectively immediately creates a change in volatility. Time-varying volatility by itself is enough to generate, qualitatively, the excess kurtosis observed in stock market returns in table 1.

## 4.1 The leverage effect

**Proposition 2** *The instantaneous coefficient in a regression of changes in the conditional variance of returns on price changes is*

$$
\frac{\operatorname{cov}\left(dp_t, dvol_t\right)}{\operatorname{var}\left(dp_t\right)} = \frac{\kappa_{3,t}}{\sigma_{Y,t}\kappa_{2,t}} \qquad (14)
$$

The term ⓐ in (13) shows that the presence of a **leverage effect** – the negative correlation between changes in volatility and prices in table 1 and the bottom panels of figure 1 – is completely determined by the third moment of agents' conditional distribution and the noise in agents' signals. The necessary and sufficient condition for the existence of a leverage effect is that $\kappa_{3,t} < 0$. There is a leverage effect if and only if agents' posterior distribution for fundamentals is negatively skewed. And the fact that we observe a leverage

[19] In principle, $\operatorname{cov}_t\left(x_t, dx_t\right)$ is related to $\kappa_{2,t}$, so it is not completely driven by fundamentals alone. However, the paper's focus will be on the case where fundamentals are a martingale, so that conditional expectations of $dx_t$ are always equal to zero, which also makes the covariance zero.

effect in the aggregate US stock market in nearly all months in the data, including during severe downturns, then implies that the conditional skewness is negative in essentially all states of the world observed in our sample. The relationship has additionally strengthened over time, which would be consistent with a decline in $\kappa_{3,t}$.

The intuition for proposition 2 is straightforward: a negative third moment means that the right tail of the conditional distribution is shorter than the left. When agents receive good news about fundamentals, that tells them they are likely on the narrower side of the distribution, and their conditional uncertainty falls, driving down return volatility.

Finally, note that $\kappa_{3,t} < 0$ does not mean that fundamentals have time-varying volatility. In the quantitative example below, that condition is satisfied even though fundamentals follow an iid process with constant volatility.

## 4.2 Slow decay in volatility

The term ⓑ in (13) shows how volatility decays. When volatility is high, $vol_t^2 \sigma_{Y,t}^{-1} dt$ also grows, pulling volatility back down towards its steady state. Interestingly, though, unlike standard time-series models (e.g. an AR(1) or Ornstein-Uhlenbeck process), the mean reversion is *quadratic*, so that the rate of mean reversion rises more than proportionately with increases in volatility.

There is a large empirical literature studying nonlinearity in volatility dynamics in securities markets. The form of mean reversion here is consistent with that literature, in that the decay is non-exponential.[20] When jumps up in $vol_t$ are large relative to its steady-state value, its decay after a time $\Delta t$ is approximately of the form $1/(1 + a\Delta t)$ for a coefficient $a$.[21] That is exactly the polynomial decay studied in the literature on long memory in volatility, and it is the inverse linear decay that is also observed in the top-right panel of figure 1.

Intuitively, volatility decays nonlinearly because the degree to which expectations respond to signals (i.e. the magnitude of the gain) is increasing in uncertainty. When uncertainty is high, agents update strongly in response to signals and learn quickly. As uncertainty falls, they update less strongly and learning slows.

[20] See Corsi (2009) for a discussion of some of the evidence (going back at least to Ding, Granger and Engle (1993)) along with the fact that the data is generally consistent both with strict long memory and also processes that simply approximate it, since formally long memory is defined asymptotically.

[21] Specifically, if there is a jump at some date $t_0$, then if there are no further shocks to volatility and $\sigma_{Y,t}$ is constant, so that it just deterministically falls, then to leading order $y_{t_0+\Delta t} \approx y_{t_0} / \left(1 + \sigma_Y^{-1} y_{t_0} \Delta t\right)$.

### 4.3 Skewness in returns

Since the price process, $p$, is a diffusion, its instantaneous skewness is not formally well defined. Skewness arises as returns interact with changes in volatility. We get the following result using a second-order approximation for the third moment of returns, defining skewness as usual as the scaled third moment:

$$skew_t\left(p_{t+\Delta t}\right) \equiv \frac{\mathbb{E}_t\left[\left(p_{t+\Delta t}-E_t p_{t+\Delta t}\right)^3\right]}{\mathbb{E}_t\left[\left(p_{t+\Delta t}-E_t p_{t+\Delta t}\right)^2\right]^{3/2}} \tag{15}$$

**Proposition 3** *The local skewness of returns is*

$$\lim_{\Delta t \downarrow 0} skew_t\left(p_{t+\Delta t}\right)\left(\Delta t\right)^{-1/2} = 3\frac{\kappa_{3,t}}{\kappa_{2,t}}\sigma_{Y,t}^{-1} \tag{16}$$

The conditional "instantaneous" skewness of returns again depends on the second and third moments of the posterior. As $\Delta t \to 0$, skewness goes to zero – that is the usual result that returns are locally normal. For small but nonzero values of $\Delta t$, equation (16) provides a link between the conditional skewness of returns – which is potentially measurable – and the conditional skewness of fundamentals, $skew_t\left(x_t\right)$, which determines the leverage effect:

**Corollary 3** *The leverage effect coefficient as defined in proposition 2 is related to the skewness of returns via*

$$\lim_{\Delta t \downarrow 0} skew_t\left(p_{t+\Delta t}\right)\frac{1}{3}\left(\Delta t\right)^{-1/2} = \frac{\operatorname{cov}\left(dp_t, dvol_t\right)}{\operatorname{var}\left(dp_t\right)} \tag{17}$$

These results show that the model is able to qualitatively match the return skewness documented in table 1 and the top-left panel of figure 1 again when $\kappa_{3,t}$ is negative. Skewness arises here again due to the term ⓐ in equation (13): declines in prices raise volatility, leading to a relatively long left tail in returns. Neuberger (2012) and Neuberger and Payne (2021) study this mechanism in detail.

### 4.4 Skewness in volatility

Table 1 shows that both the level and daily changes in stock market volatility are also skewed. The source of that effect can be seen by combining equations (8) and (16) to obtain

$$std\left(vol_t\right) = \frac{1}{3}vol_t\left|\lim_{\Delta t \downarrow 0} skew_t\left(p_{t+\Delta t}\right)\left(\Delta t\right)^{-1/2}\right| + o\left(\Delta t^{1/2}\right) \tag{18}$$

Holding the conditional skewness of returns fixed, the volatility of innovations to $vol_t$ scales with $vol_t$ itself. Past work (e.g. Bollerslev, Tauchen and Zhou (2009)) has emphasized the importance of time-varying vol-of-vol. This present model gets it through an endogenous mechanism. Note also that this variation does not just come from an assumption that the volatility of fundamentals following a nonlinear process, as in Cox, Jr. and Ross (1985). Finally, as with returns, time-varying volatility in volatility mechanically also generates skewness (in this case positive) and excess kurtosis in the unconditional distribution of volatility.

## 4.5 Examples

This section briefly considers two simple examples. Section 6 studies in much more depth a quantitatively realistic example.

### 4.5.1 Linear Gaussian process

If fundamentals, $x$, follow a linear Gaussian process then the model's solution is the Kalman filter. $p_t$ is a linear function of the history of signals; its gain and hence conditional variance converge to constants; and its conditional skewness and all higher cumulants are always equal to zero. There is then no leverage effect, volatility of volatility, or skewness in prices or their volatility.

### 4.5.2 Markov switching process

Veronesi (1999) studies a two-state switching model in which the latent state $x$ switches between a low and a high value at rates $\lambda_{HL}$ and $\lambda_{LH}$, respectively, and agents have a Gaussian signal as required in proposition 1. In this case, the low and high values of $x_t$ can be normalized to 0 and 1 without loss of generality.

Agents' beliefs at any given time are summarized by single parameter, $\pi_t$, which is their posterior probability that $x_t = 1$. The conditional variance and third moment of $x_t$, which drive price dynamics, are simple functions of $\pi_t$:

$$\kappa_{2,t} = \pi_t (1 - \pi_t) \tag{19}$$

$$\kappa_{3,t} = (1 - 2\pi_t) \times \kappa_{2,t} \tag{20}$$

The variance here then is a bell-shaped function of $\pi_t$, peaking at 1/4 at $\pi_t = 1/2$ and declining to zero on both sides, and $sign(\kappa_{3,t}) = sign\left(\frac{1}{2} - \pi_t\right)$. Economically, when $\pi_t$ is near 1 so that agents are confident they are in the good state, volatility is low, but the third moment is strongly negative, so there is a leverage effect. However, when a bad state is realized and investors have seen enough signals to be confident in that, so that $\pi_t$ is below 1/2, volatility falls and the leverage effect *reverses*: agents no longer worry as much about the economy getting worse, so there is relatively more upside and $\kappa_{3,t} > 0$.

These results illustrate the importance, in the context of the leverage effect, of agents continuing to learn in bad states. If learning effectively stops once agents know the economy is in a recession – in the sense that things cannot continue to get worse – then the leverage effect disappears or even reverses. That is not what is observed historically in the US stock market.

# 5 Extensions and robustness

## 5.1 How much rationality needs to be assumed here?

At first glance, this model appears to require investors to be strongly rational, applying Bayes theorem with full knowledge of the dynamics driving fundamentals. While that is the baseline in the simulations below, there is nothing about the analysis that requires it. Proposition 1 and remark 1 hold as long as agents update using the basic *form* of Bayes' rule. In particular, none of the following is required in the derivations:

1. That agents use the correct precision ($\sigma^2_{Y,t}$) in calculating their update (e.g. the diagnostic expectations of Bordalo, Gennaioli, Porta and Shleifer (2019))
2. That agents' assumed dynamics for $x$ are in any sense "correct"
3. That the signals agents observe are truly Gaussian
4. That the agents incorporate all information they receive. They could irrationally or inefficiently ignore some information, and unreasonably privilege other sources
5. That the agents properly weight all information they receive. Agents might receive many Gaussian signals, which can be combined into a single value and used for updating. It is possible that they do that combination incorrectly

What is important here is not actually that agents are true Bayesians. The propositions and corollaries above simply require that they use an updating rule that has the same algebraic structure as Bayes' rule. Obviously deviations from rational expectations will mean that what agents see as a martingale will not appear to be one to an econometrician. But the nonlinearities the paper focuses on do not hinge on anything like complete rationality. Molavi (2025) discusses these issues in greater detail.

## 5.2 Discrete information revelation events

Dytso, Poor and Shitz (2022) prove the following discrete version of theorem 1. Instead of assuming a diffusive information flow, this result is for a signal with strictly positive information content, meaning that the moments also update by discrete amounts.

**Proposition 4** *[Dytso, Poor and Shitz (2022), equation (52)] For a random variable $x_t$ and a signal $y_t \sim N(x_t, \sigma^2)$,*

$$\frac{d}{dy}\kappa_j(x_t \mid y_t = a) = \kappa_{j+1}(x_t \mid y_t = a)/\sigma^2 \tag{21}$$

*where $\kappa_j(x_t \mid y_t = a)$ is the $j$th posterior cumulant of $x_t$ conditional on observing $y_t = a$. Furthermore, for $y_t$ in a neighborhood of any $a \in \mathbb{R}$,*

$$\mathbb{E}(x_t \mid y_t) = \sum_{j=0}^{\infty} \frac{\kappa_{j+1}(x_t \mid y_t = a)}{j!}\left(\frac{y_t - a}{\sigma^2}\right)^j \tag{22}$$

Proposition 4 shows that the type of recursion in theorem 1 also holds for discrete revelation events – diffusive information is not necessary for the central results. At the same time, it shows that normality is important – continuity is not essential, but proposition 4 still requires a normally distributed signal.[22] That said, proposition 4 also shows why continuous time is useful here: in (21) the *posterior* cumulants, which are precisely what one wants to solve for, determine sensitivity. In continuous time the cumulants follow continuous processes, so the prior and posterior values are effectively identical, which simplifies the analysis. Additionally, though, note that theorem 1 does not follow directly from proposition 4. Accounting for dynamics, which is ultimately central to the analysis, makes the derivations significantly more complicated.

[22] Dytso and Cardone (2021) explore related results for non-Gaussian variables, but do not derive a power series result. It is possible to derive a similar result for certain other specific cases, e.g. when the likelihood is exponential or Poisson.

### 5.3 Allowing marginal utility or cash flows to depend on information

The main results restrict to the case where state prices are not affected by the realization of the signal that agents observe. In some of the learning models in the literature, there is an additional mechanism that enters because information is priced (i.e. $M$ depends on the signal itself). This paper rules that channel out because it adds significant complications to the analysis and the goal is to focus on the implications that follow directly the learning, rather than additional price effects.

In models in which information is endogenously priced, such as the CAPM and generalized recursive preferences, that price is not a known exogenous function. Rather, appendix A.3.5.2 shows that finding it involves solving a fixed point problem. In the CAPM for example, the response of prices to a shock depends partly on how future volatility responds. But the response of future volatility itself depends on how prices respond to shocks.[23] That fixed point renders the model unsolvable by hand, and the endogeneity of risk premia is not necessary for matching the features of the data that motivate this paper's analysis. There is nothing about that fixed point structure, though, that appears impossible in principle to incorporate into this paper's type of analysis in future work, but it would likely need to be solved numerically (as is the usual practice in the literature on learning).

That said, appendix A.3.6 shows that in an extension of the model to capture the CAPM idea that volatility drives risk premia, the empirically relevant case of $\kappa_3 < 0$ also generally predicts that prices will display excess volatility and countercyclical expected returns, as in the data.

## 6 Illustrative calibration

This section presents a simple quantitative example. We first use it to illustrate the model's core mechanisms with impulse response functions and then examine the extent to which the qualitative predictions above can potentially map into quantitatively reasonable behavior.

[23] The information structure from Veronesi (1999) that is studied in section 4.5.2 is a special case of what is studied here, but the full model in Veronesi (1999) involves exactly this fixed point, which in the end is solved numerically.

## 6.1 Model setup

Fundamentals have an average growth rate of $g$ with both small Gaussian shocks and occasional downward jumps:

$$dx_t = (\phi\lambda + g)\, dt - J_t dN_t + \sigma_x dB_t \tag{23}$$

where $B$ is a Brownian motion, $N$ is a Poisson process with constant rate $\phi$, and $J_t$ is a random variable with mean $\lambda$. The term $\phi\lambda dt$ ensures that mean price growth is equal to $g$.[24] The drift $g$ can be thought of (and formally motivated as) coming from a risk premium on cash flows. It plays no role other than to generate positive average returns.

In the absence of learning, prices are equal to $x$ and hence inherit its dynamics. So without learning, returns would be skewed due to the jumps, but volatility would be constant and there would be no leverage effect or long memory.

We calibrate the jump distribution based on Barro and Jin (2011) (though with smaller disasters assumed to have somewhat heavier tails) and assume that equities have a leverage of 3 relative to shocks to consumption.[25] $\sigma_x$ is then chosen to approximately match the standard deviation of US stock returns, and $\sigma_Y = 6.5$ (based on the time index corresponding to days) was chosen relatively roughly to match the dynamics of volatility and skewness. To get a sense of scale, if agents hypothetically had a prior variance for fundamentals of $\infty$ and fundamentals were constant, after one year of observing such signals their posterior standard deviation would be 0.41 in logs. Since $\sigma_Y$ controls the rate of learning, it determines both the autocorrelations of volatility and also the time from peak to trough following downward jumps in fundamentals. We compare both to the data below to help validate the choice of $\sigma_Y$. Finally, we set $g = 0.07$ to match the historical equity premium.

It is straightforward to simulate the model by discretizing the state space and then calculating expectations using Bayes' theorem. In addition, prices are calculated as $P_t = \mathbb{E}_t \exp(x_t)$, so that the simulation does not use the log approximation from equation (4).

[24] The reduced-form process for $x$ can easily be generated by assuming that cash-flows follow the same jump process. Positive risk premia can be generated by assuming the SDF is also driven by the same jumps (but with the opposite exposure).

[25] Leverage of 3 is high relative to the average debt-to-equity ratio for US stocks. On the other hand, large negative shocks mechanically raise leverage, so a value of 3 can be seen as representing more how leverage would act in a disaster state. This value is consistent with that chosen by Bansal and Yaron (2004).

The exact calibration is the double power law with parameter estimates from consumption disasters in the top panel of their table I, with the modification that the parameter $\beta$ is set to 5.

## 6.2 Impulse response functions

This section examines two impulse response functions – to errors in the signal, $\sigma_Y dW_t$, and to jumps in fundamentals, $J_t dN_t$ – calculated as the population mean (from a 100,000-year simulation) conditional on a shock having occurred compared to the population mean conditional on a shock not having occurred.

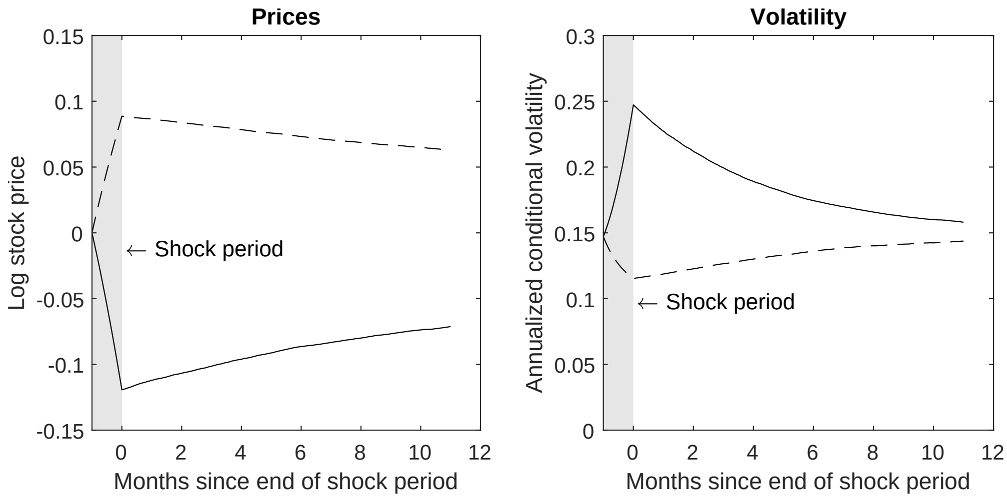
The noise shock is defined as a month with a one-in-ten-year (±2.4 standard deviation) realization in the total error in the signal over the month. Figure 2 plots the response of prices and volatility to the shock. The first month in the figure is the period in which the shock occurs. In the case of a negative shock, prices fall and uncertainty rises. As uncertainty (and hence volatility) rises, prices become more sensitive to signals, with the result that the response of prices over the course of the month is concave, with prices declining progressively faster. When the shock ends, prices and volatility revert, but the recovery is very slow: volatility only gets close to its starting point after about a year, and in the same time prices have recovered only about half of their decline. The fact that prices recover initially quickly and then slowly is again a consequence of the volatility dynamics: the recovery of volatility and uncertainty itself slows the recovery in prices because when uncertainty is low, agents update less in response to the incoming data that is (by assumption) telling them that a disaster did not actually happen.

In the case of a positive noise shock, the effects are smaller because of the asymmetry in the fundamentals process. Agents understand that it is possible that a large negative shock has occurred, so they update strongly in response to negative signals, but since large positive shocks are effectively impossible, they strongly discount positive signals. That plays out by volatility declining, so that the marginal effects of additional noise are progressively smaller. The overall effect on prices of a negative noise shock is 46% larger than a positive noise shock, and the effect on volatility is 2.9 times larger.

The second pair of IRFs, in figure 3, represents the average response to a downward jump in fundamentals. The left-hand plot in figure 3 shows that the decline in prices is again nonlinear – it accelerates before slowing, with the initial declines on average equal to 0.13 percent per day, accelerating to 0.22 percent per day at their peak. It takes at least five years on average for prices to fully incorporate the drop in fundamentals. This is therefore a model in which disasters take years on average to fully play out. This behavior is similar to that studied in Ghaderi, Kilic and Seo (2022), with the difference here being that it happens endogenously, instead of there being an assumption that disaster shocks are clustered.

To get a sense of whether the length of crashes is empirically reasonable, figure A.1 in the appendix plots the time from peak to trough for all drawdowns in the US stock market

Figure 2: Response to negative error in the signal




**Note:** The left-hand panel plots the IRF for prices – the conditional expectation of $x$ – and the right hand for the conditional standard deviation of prices. The shock is a one-time unit standard deviation negative error in the signal (i.e. a negative realization of $\sigma_Y dW$).

greater than 10 percent compared to the average time for drawdowns of the same magnitude in the simulations. If anything, the model actually appears to *understate* the time from peak to trough relative to the data, implying that learning in the data is even slower than what is calibrated here.

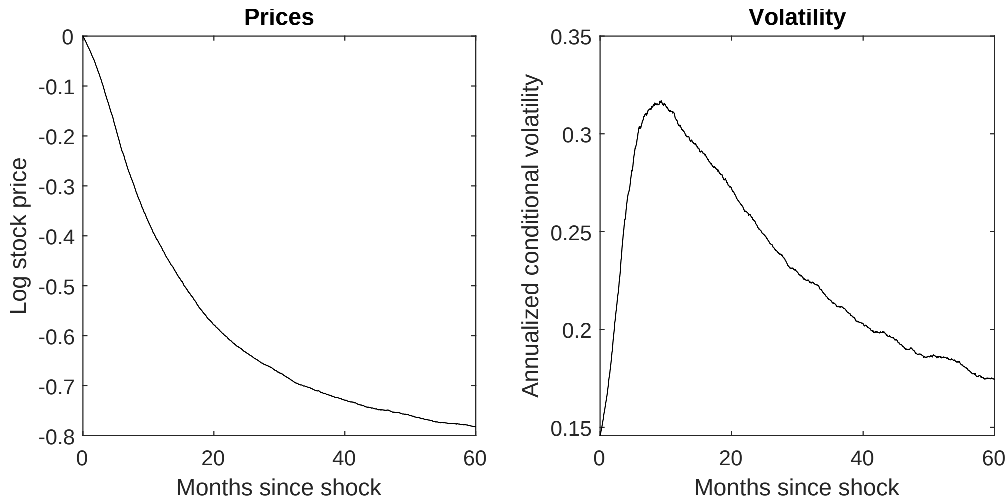
The right-hand panel of figure 3 shows that the peak in volatility does not come on average until 9 months after the shock. That said, because of the nonlinearity of the model, the mean IRF is not a very good representation of typical behavior. For example, the average *maximum* of annualized volatility in the five years after a negative jump is 80 percent, more than twice as high as the average five-year peak of only 37 percent for periods with no jump.

## 6.3 Simulation results

The left-hand panel of figure 4 plots an example of a 100-year time-series of fundamentals, $x$, and prices, $p$, from the full simulation. Prices track fundamentals well in the long-run, but clearly there can be large temporary deviations, and those deviations are skewed left. In some cases, fundamentals jump down and it takes time for prices to catch up, and there are also clear cases of prices jumping down "erroneously" (based on hindsight or on knowing the true state) and then recovering, for example around year 50.

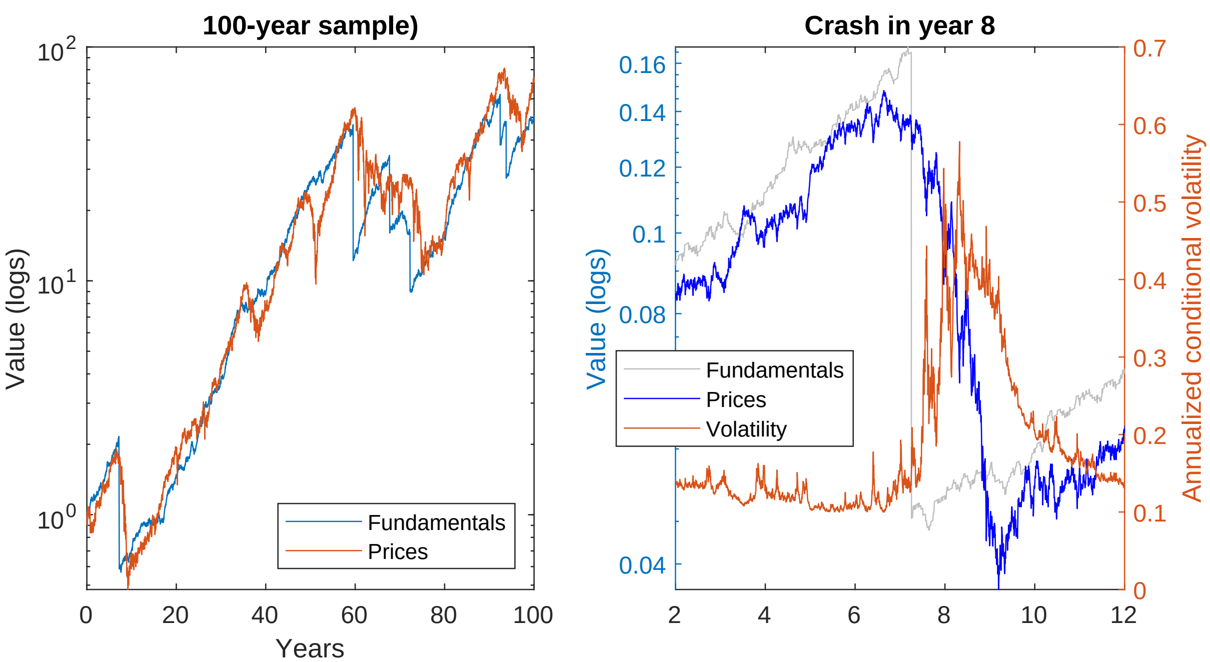
The right-hand panel of figure 4 plots the dynamics of prices and conditional volatility during the crash observed around year 8 in the simulation in the left-hand panel. The crash

Figure 3: Response to a negative realization of fundamentals




**Note:** These plots are the same as in the previous figure, except they correspond to the IRF for a negative realization of fundamentals. Specifically, the IRF for prices is the average path of prices when $x = -\lambda$ compared to $x = 0$, and the right-hand panel is the same for price volatility.

Figure 4: Simulated time series of fundamentals and prices




**Note:** "Fundamentals" is the simulated $x$ process, and "prices" is the simulated $\kappa_1$ process. The mean growth rate has been removed to help make the figure readable.

displays what can be recognized as common behavior in the data, with the price decline accelerating and then ending with a large rebound from the bottom. Volatility rises as prices fall, peaking near the bottom, and then falling as prices recover somewhat.

Table 2 reports moments for returns and their conditional volatility. As discussed in Barro and Jin (2011), the US has historically had fewer disasters than would be expected unconditionally. We therefore calculate moments as the average from 100-year samples of the simulation in which the most negative annual return is no more negative than the most negative value observed in the US over the past century. That choice leads to somewhat less skewness in the simulations. Calculating moments on 100-year samples also accounts for any small-sample bias in the statistics.

Table 2: Model vs data moments

| | Stock returns | | Volatility level | | Volatility change | |
|---|---|---|---|---|---|---|
| Moment | Data | Model | Data | Model | Data | Model |
| Std. dev. | 1.14 | 0.98 | 6.73 | 6.72 | 1.43 | 0.88 |
| Skewness | -0.06 | 0.02 | 2.16 | 4.25 | 1.37 | 0.15 |
| Kurtosis | 19.54 | 12.14 | 11.38 | 33.35 | 30.36 | 75.88 |
| Corr. w/ $R_t$ | | | | | -0.78 | -0.76 |

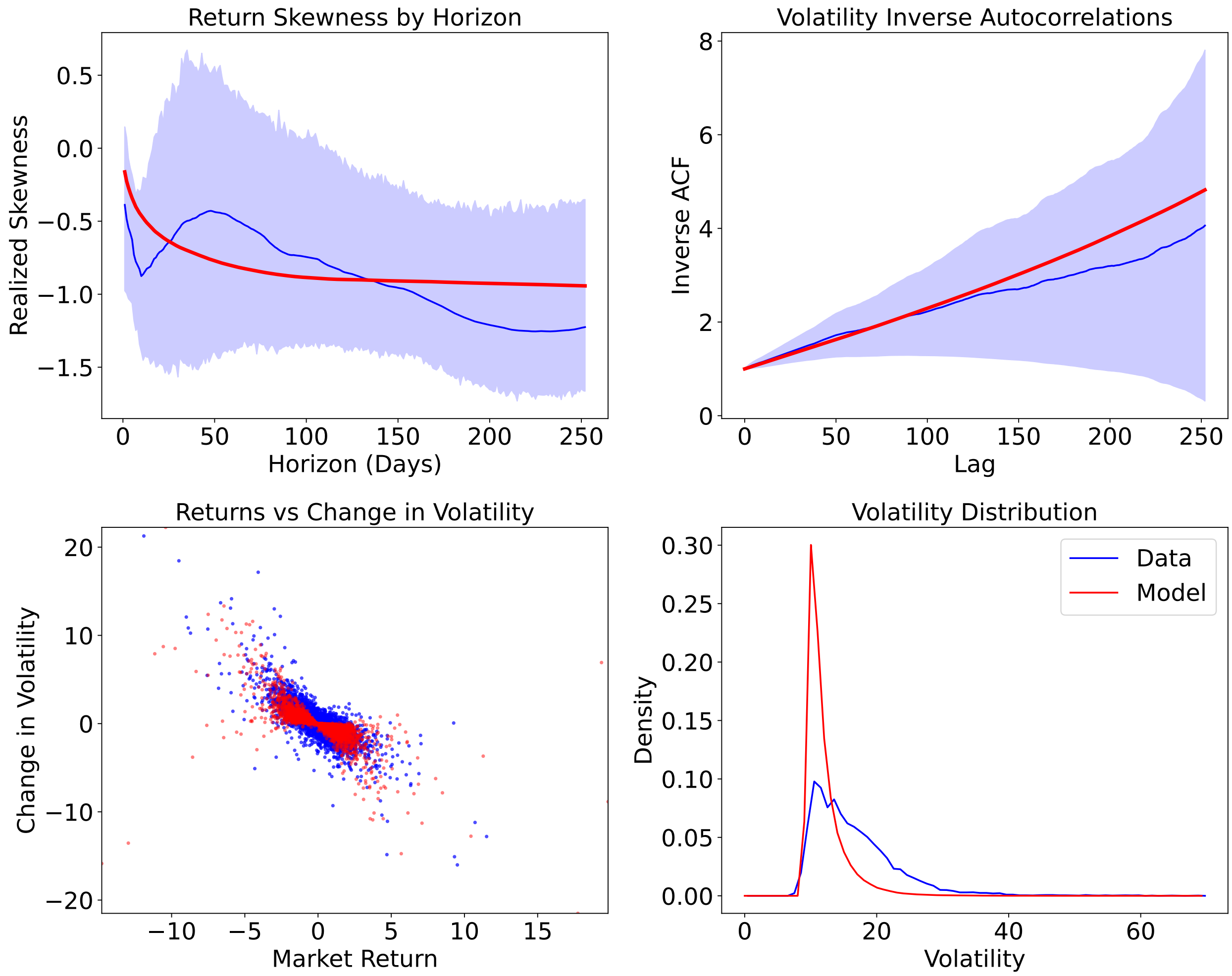
**Note:** This replicates table 1 for the empirical moments and compares them to the model simulation.

Table 2 shows that model broadly matches the data. Kurtosis of stock returns is a bit smaller than in the data, but that statistic is estimated with large confidence bands. Volatility in the model has a similar unconditional volatility to the data, but with greater skewness and kurtosis. The correlation of changes in volatility and returns is nearly identical to the data.

Figure 5 compares the model's behavior to what was shown in figure 1. Log return skewness and volatility autocorrelations are both highly similar to the data at all horizons. The bottom-left panel shows that the model generates a leverage effect scatter plot similar to the data, though with too little dispersion when returns are near zero, implying that there is a component of conditional volatility that is driven by a process that is independent of returns. Finally, the bottom-right panel shows histograms for volatility in the model and data demonstrating that the distributions are reasonably similar, though, as is also reflected in the table, the skewness in the simulation is much higher than the data.

The results in this section show that the model is able to match key features of the data not just qualitatively but also quantitatively. The overall volatility of returns and the fact that they are negatively skewed comes from the calibration of fundamentals. But the volatility dynamics and the evolution of skewness across horizons are both absent from fundamentals and arise from the learning mechanism.

Figure 5: Simulation results




**Note:** The top left, top right and bottom left panels are the same as in figure 1, overlayed with the corresponding outputs from a 100-year simulation of the model based on the parameters discussed in the text. The bottom-right panel shows the estimated density of volatility in the data and in the model.

# 7 Estimated volatility dynamics and investor uncertainty

The analytic results in section 4 have specific implications for the dynamics of volatility and the leverage effect. This section focuses on estimating the regressions motivated by the model-implied dynamics for volatility.

## 7.1 Regression setup

Combining equations (12), (8), and (16), and assuming the price is a martingale and holding $\sigma_{Y,t}$ constant, we have

$$d\left(vol_t\right) = \left(\frac{1}{3}\Delta t^{-1/2}\right) skew_t\left(p_{t+\Delta t}\right) dp_t - \frac{1}{\sigma_{Y,t}} vol_t^2 dt + \mathbb{E}_t\left[d\left\langle x\right\rangle_t\right] \tag{24}$$

If $x$ has independent increments – as in the quantitative model – then the last term reduces to a constant. We take that as our benchmark in this section.

The model has two predictions for the results of this regression: the coefficient on $\left(\frac{1}{3}\Delta t^{-1/2}\right) skew\left(p_{t+\Delta t}\right) dp_t$ should be equal to 1, and the coefficient on $vol_t^2 dt$ is equal to $\sigma_{Y,t}^{-1}$. The first relationship holds as long as prices and volatility follow a joint diffusion driven by a single Brownian motion, and so it tests that aspect of the model. The second prediction shows that the regression can be used to identify one of the model's structural parameters. Additionally, we test the model's prediction that the mean reversion is quadratic rather than linear.

### 7.2 Data

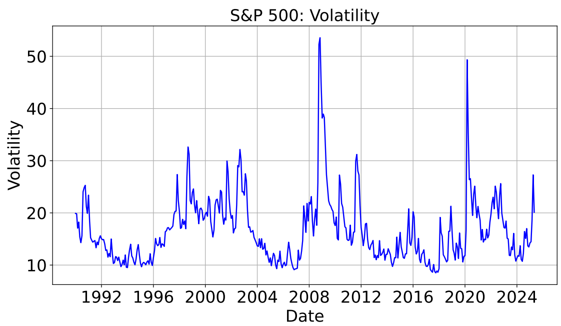
We estimate the regression (24) for two markets. The first, given our focus on the stock market, is the S&P 500. For that case, we proxy for $vol_t$ with the same conditional volatility measure that we have used throughout. For the return $dp_t$ we continue to use the log return on the CRSP total market index. Last, similar to volatility, we construct $skew_t\left(p_{t+\Delta t}\right)$ by directly forecasting the realized second and third moments using option-implied moments. The top-left panel of figure 6 plots the measure of conditional volatility and the bottom-left conditional skewness over our sample period.

The S&P 500 conditional skewness is almost exclusively negative, so for the second market to use for estimation, we choose natural gas because it is a large and economically significant market that displays, in contrast, consistently positive skewness. To calculate $dp_t$ in this case we use natural gas futures returns from the CME. We then use options on futures to estimate the conditional moments in the same way as for the S&P 500.[26] The time series of conditional volatility and skewness for natural gas are plotted in the right-hand panels of figure 6. Due to the seasonality in natural gas prices, we include contract fixed effects in the volatility regressions for natural gas.

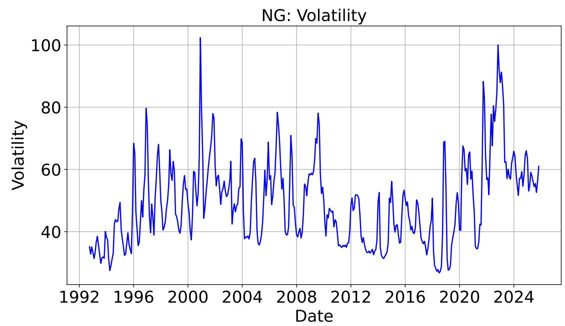
There are many other underlyings that could be studied, like individual stocks, bonds, and other commodities. The two markets here are simply meant to illustrate the model's core mechanisms and show that in at least two notable cases they map somewhat reasonably into the data.

[26] The specific methods are from Dew-Becker (2024).

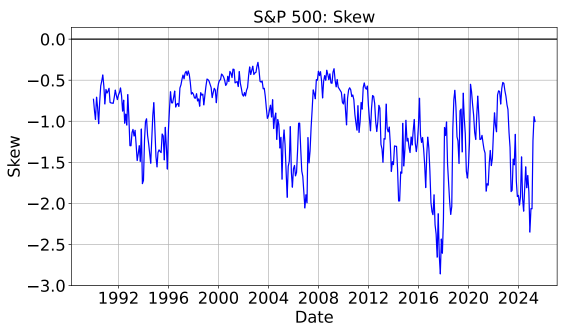
Figure 6: Time series of VIX and Skew for S&P 500 and natural gas



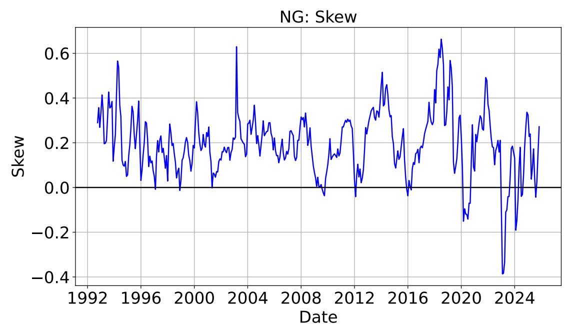

(a) Volatility (S&P 500)




(b) Volatility (natural gas)




(c) Skew (S&P 500)




(d) Skew (natural gas)

**Note:** Time series plots for the volatility and skewness for the S&P 500 and for natural gas. All series are obtained from regressing realized moments onto the corresponding option-implied moments, and using fitted values to account for risk premia in options. All series are monthly averages of the corresponding daily series.

## 7.3 Results

Table 3 reports results of the regression implied by (24). The first and third columns report the baseline results. The coefficients are highly statistically significant and have the expected signs.

Under the model, if the various assumptions made to derive the regression setup are true, the coefficient on $\frac{1}{3\sqrt{21}} skew_t dp_t$ should be equal to 1, and in both cases that is a very good description of the data. The coefficient is 1.01 with a standard error of 0.03 for the S&P 500 and 1.13 with a standard error of 0.06 for natural gas. There are really two key features of the model that generate that prediction: that returns and their volatility are jointly driven by the same shock (i.e. the same Brownian motion), and that they jointly follow a diffusion.

Table 3: Volatility regressions

| | S&P 500 | | | Natural Gas | | |
|---|---|---|---|---|---|---|
| | (1) | (2) | (3) | (4) | (5) | (6) |
| $Vol_{t-1}^2$ | -0.7449*** | -0.7611 | -0.5712*** | -0.1496*** | -0.6816*** | -0.1497*** |
| | [0.1605] | [0.5096] | [0.1355] | [0.0416] | [0.2556] | [0.0415] |
| $\frac{1}{3\sqrt{21}} skew_{t-1} dp_t$ | 1.0078*** | 1.0077*** | 0.2784*** | 1.1333*** | 1.1303*** | 1.1140*** |
| | [0.0265] | [0.0265] | [0.0410] | [0.0625] | [0.0624] | [0.0831] |
| $Vol_{t-1}$ | | 0.0005 | | | 0.0389** | |
| | | [0.0124] | | | [0.0165] | |
| $dp_t$ | | | -0.0477*** | | | 0.0005 |
| | | | [0.0028] | | | [0.0012] |
| $R^2$ | 0.5489 | 0.5489 | 0.6306 | 0.1835 | 0.1856 | 0.1836 |

**Note:** Daily regressions of first differences in volatility (for S&P 500 and natural gas) onto different predictors. Stars indicate statistical significance: * p<.1, ** p<.05, ***p<.01.

To evaluate the relevance of the model-implied nonlinearity in volatility dynamics, the second and fourth columns of table 3 include both $vol_{t-1}^2$ and $vol_{t-1}$. $vol_{t-1}^2$ does in fact appear to dominate. In both cases, the t-statistic for $vol_{t-1}^2$ is larger than that on $vol_{t-1}$ indicating that $vol_{t-1}^2$ has greater explanatory power than $vol_{t-1}$. For the S&P the t-statistic for $vol_{t-1}^2$ is larger than that for $vol_{t-1}$by a factor of nearly 40, but it is only marginally larger for natural gas.

## 7.4 Estimates of investors' uncertainty about fundamentals

The coefficients on $vol_{t-1}^2$ give an estimate of $\sigma_Y^{-1}$ at the daily level (given that these are daily regressions). Having estimates for $\sigma_Y$ allows us to then use the volatility and skewness of stock market returns to reveal the standard deviation and skewness of agents' posteriors for fundamentals. Specifically, recall that

$$vol_t = \frac{\kappa_{2,t}}{\sigma_{Y,t}} \Delta t^{1/2} \tag{25}$$

$$\Rightarrow \kappa_{2,t}^{1/2} = \left(vol_t \sigma_{Y,t} \Delta t^{-1/2}\right)^{1/2} \tag{26}$$

and that the scaling of the estimates is for a unit time interval being equal to a day.

$\sigma_{Y,t}$ is between 0.94 and 2.36, based on the coefficient on $vol_{t-1}^2$ in equation (24). The first thing to note is that that value is lower than the value that works well in the calibration studied above, showing that the calibration is at least somewhat misspecified for volatility dynamics.

Taking equation (26) and inserting the value for $\sigma_Y$ along with the historical daily standard deviation of stock returns in our sample, 1.05 percent, implies that agents' posterior standard deviation is between 10.4 and 16.5 percent. The $\pm 2$ standard deviation range for fundamentals around the current price for the aggregate stock market is then between $\pm 20.8$ and $\pm 33.0$ percent.

Similarly, we can get an estimate of average skewness in beliefs. One-month conditional return skewness is historically approximately -1. Plugging that into (16) along with the estimates of $\kappa_2$ and $\sigma_Y$ yields an estimate for the skewness of fundamentals between -0.29 and -1.13. In the time series, the estimate of conditional skewness of fundamentals is proportional to the conditional skewness of returns divided by the square root of the conditional standard deviation of returns.

These estimates are both notable because they are independent of the model for $x$ – in that sense they are model free. What they depend on is just the information structure the paper assumes, which is that prices are driven by a single composite signal that is Gaussian conditional on $x$.

## 7.5 Comparing to survey data

We have not found a survey that directly measures investors' uncertainty about fundamentals and would allow us to validate the estimate of $\sigma_Y$ – i.e. a survey that asks about probabilities that the fundamental value might fall in different ranges. However, uncertainty is sometimes proxied for by disagreement, and it is plausible that they are at least somewhat related.

The *Investor Behavior Project* at Yale has a survey of institutional investors that asks the following question: "What do you think would be a sensible level for the Dow Jones

Industrial Average based on your assessment of U.S. corporate strength (fundamentals)?" We interpret the answer to that question as each investor's estimate of $E\left[\exp\left(x_t\right) \mid Y^t\right]$. To calculate cross-sectional dispersion, given that the surveys are completed on different dates by different respondents, we calculate the average squared log difference between each investor's reported fundamental value and the actual value at the time of the survey. The square root of that average represents a measure of the cross-sectional standard deviation.

The data runs from August 1993 to July 2024 and has 8,242 observations. In that sample, the cross-sectional standard deviation is 17.0 percent, which lies just outside the edge of the confidence bands for $\kappa_2^{1/2}$. Again, while disagreement and uncertainty are different concepts, it is plausible that degree of disagreement across people would be of a similar magnitude to overall uncertainty, and we observe that here.

### 7.6 Summary

Overall, this section shows that the model's predictions for volatility dynamics match the data well, both for the S&P 500 and natural gas futures. The prediction for nonlinear mean reversion – via a quadratic term in the regression – is well confirmed, and in fact it drives out a linear mean reversion term. The model's prediction of a coefficient of 1 on the correctly scaled interaction of returns with conditional skewness also fits well, showing that conditional skewness controls the magnitude of the leverage effect over time. Finally, the coefficients themselves can be mapped into an estimate of $\sigma_Y$, the noise in investors' signals.

## 8 Conclusion

This paper's main results are fundamentally about how information affects beliefs in a very simple but standard Bayesian filtering setting. The analysis is motivated by the highly nonlinear behavior of the stock market, and it shows that belief dynamics under information acquisition explain many of those nonlinearities, both qualitatively and quantitatively. While many of the results here have been shown to hold in specific learning settings, the contribution of this paper is to show that they are generic predictions and to give necessary and sufficient conditions for the model to match features of the data like the leverage effect.

The results are much more broadly applicable, though. Obviously there are many other financial markets that display different forms of nonlinearity, and it is natural to ask how well the filtering mechanism works in those settings. Discrete information revelation events play no role in the analysis of this paper, but are certainly much more important for individual stocks.

But information acquisition problems are pervasive in economics, and the assumptions of linearity and Gaussianity are not always reasonable approximations to the data. Inflation, for example, has historically been highly skewed, with its volatility being positively correlated with its level. The analysis here may therefore be useful in understanding the evolution of inflation expectations in the face of that nonlinearity.

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# A.1 Proof of theorem 1

## A.1.1 Assumptions

**Assumption 3** *Let $\phi_{\omega,t} = \exp(i\omega x_t)$ denote the complex exponential of $x_t$. For any $\omega$, there exists an adapted process $\mathcal{G}_{\omega,s}$ satisfying, a.s., $\int_0^t |\mathcal{G}_{\omega,s}|ds < \infty$ and $\int_0^t \mathbb{E}[\mathcal{G}_{\omega,s}^2]ds < \infty$, such that $\phi_{\omega,t} - \phi_{\omega,0} - \int_0^t \mathcal{G}_{\omega,s}ds$ is a right-continuous martingale.*

**Assumption 4** $\int_0^t \mathbb{E}[x_s^2]ds < \infty$ *and* $\int_0^t |x_s|ds < \infty$ *almost surely.*

**Assumption 5** *The process $\sigma_{Y,t}$ is progressively measurable with respect to the natural filtration of $Y_t$. Furthermore,*

$$\mathbb{P}\left(\int_0^t \sigma_{Y,s}^2 ds < \infty\right) = 1, \tag{A.1}$$

$$0 < \underline{\sigma}^2 \leq \sigma_{Y,t}^2, \tag{A.2}$$

$$\left|\sigma_{Y,t} - \sigma_{\tilde{Y},t}\right|^2 \leq L_1 \int_0^t (Y_s - \tilde{Y}_s)^2 dK(s) + L_2(Y_t - \tilde{Y}_t)^2, \tag{A.3}$$

$$\sigma_{Y,t}^2 \leq L_1 \int_0^t (1 + Y_s^2)dK(s) + L_2(1 + Y_t^2), \tag{A.4}$$

*where $L_1$ and $L_2$ are non-negative constants and $K(t)$ is a non-decreasing right-continuous function satisfying $0 \leq K(t) \leq 1$ for all $t < \infty$.*

## A.1.2 Proof

**Lemma 1** *Let $\varphi_{x,t}(\omega) = \mathbb{E}[\exp(i\omega x_t)|Y^t]$ denote the characteristic function of the posterior distribution of $x_t$ conditional on $Y^t$. If assumptions 3–5 are satisfied, then*

$$d\varphi_{x,t}(\omega) = \mathbb{E}_t[d\exp(i\omega x_t)] + \mathrm{cov}_t(x_t, \exp(i\omega x_t))\frac{dY_t - \mathbb{E}_t[x_t]dt}{\sigma_{Y,t}^2}, \tag{A.5}$$

*where* $\mathbb{E}_t$ *and* $\mathrm{cov}_t$ *denote the expectation and covariance operators, respectively, conditional on* $Y^t$.

The lemma follows from theorem 8.1 of Liptser and Shiryaev (2013) by setting $h_t \to \phi_{\omega,t}$, $\xi_t \to Y_t$, $A_t \to x_t$, and $B_t(\xi) \to \sigma_{Y,t}$.[1] We proceed by verifying that conditions (8.1)–(8.9) of Liptser and Shiryaev (2013) are satisfied.

Equation (8.2) is simply equation (6) of the paper in integral form. Assumption 3 implies that conditions (8.1) and (8.7) are satisfied. The first part of condition (8.3) and condition (8.8) are satisfied by assumption 4. The second part of condition (8.3) and conditions (8.4), (8.5), (8.9) are satisfied by assumption 5. Condition (8.6) is satisfied since $\phi_{\omega,t}$ is a bounded function. Finally, applying theorem 8.1 and noting that the Brownian motion $W_t$ is independent of $x_t$, we get

$$\mathbb{E}_t[\exp(i\omega x_t)] = \mathbb{E}_0[\exp(i\omega x_0)] + \int_0^t \mathbb{E}_s[\mathcal{G}_{\omega,s}]ds + \int_0^t \frac{\mathrm{cov}_s\left(x_s, \exp(i\omega x_s)\right)}{\sigma_{Y,s}} d\overline{W}_s, \tag{A.6}$$

where

$$\overline{W}_t = \int_0^t \frac{dY_s - \mathbb{E}_s[x_s]ds}{\sigma_{Y,s}}. \tag{A.7}$$

Or equivalently,

$$d\mathbb{E}_t[\exp(i\omega x_t)] = \mathbb{E}_t[\mathcal{G}_{\omega,t}]dt + \mathrm{cov}_t(x_t, \exp(i\omega x_t))\frac{dY_t - \mathbb{E}_t[x_t]dt}{\sigma_{Y,t}^2}. \tag{A.8}$$

On the other hand, by the definition of $\mathcal{G}_{\omega,t}$,

$$d\exp(i\omega x_t) - \mathcal{G}_{\omega,t}dt = dM_t, \tag{A.9}$$

where $M_t$ is a right-continuous martingale. Therefore, $\mathbb{E}_t[\mathcal{G}_{\omega,t}]dt = \mathbb{E}_t[d\exp(i\omega x_t)]$.

**Theorem 4** *Let* $\kappa_{k,t}$ *denote the kth cumulant of the posterior distribution of* $x_t$ *conditional on* $Y^t$. *Suppose the* $n+1$*th moment of the posterior distribution and* $\mathbb{E}_t[d(x_t^n)]$ *both exist,*

[1] The result is stated for real-valued functions. However, it can trivially be extended to the complex-valued function $x \mapsto \exp(i\omega x)$ using the identity $\exp(i\omega x) = \cos(\omega x) + i\sin(\omega x)$ and separately considering the real and imaginary parts of the function.

*and assumptions 3–5 are satisfied. Then for every $k \leq n$,*

$$d\kappa_{k,t} = \sum_{j=1}^{k} \binom{k}{j} B_{k-j}\left(-\kappa_{1,t}, \ldots, -\kappa_{k-j,t}\right) \mathbb{E}_t[d(x_t^j)] + \frac{\kappa_{k+1,t}}{\sigma_{Y,t}^2}\left(dY_t - \mathbb{E}_t[x_t]dt\right)$$
$$- \frac{1}{2\sigma_{Y,t}^2} \sum_{j=2}^{k} \binom{k}{j-1} \kappa_{j,t}\kappa_{k-j+2,t}dt, \tag{A.10}$$

*where $B_j$ denotes the $j$th complete exponential Bell polynomial.*

The result follows from applying Itô's lemma to lemma 1, yielding

$$d \log \varphi_{x,t}(\omega) = \frac{\mathbb{E}_t[d\exp(i\omega x_t)]}{\mathbb{E}_t[\exp(i\omega x_t)]} + \frac{\operatorname{cov}_t(x_t, \exp(i\omega x_t))}{\mathbb{E}_t[\exp(i\omega x_t)]} \frac{dY_t - \mathbb{E}_t[x_t]dt}{\sigma_{Y,t}^2}$$
$$- \frac{1}{2\sigma_{Y,t}^2}\left(\frac{\operatorname{cov}_t(x_t, \exp(i\omega x_t))}{\mathbb{E}_t[\exp(i\omega x_t)]}\right)^2 dt \tag{A.11}$$

and then taking derivatives of both sides.

We begin by verifying existence. By assumption, the posterior distribution of $x_t$ conditional on $Y^t$ has $n+1$ moments. Therefore, the posterior also has $n+1$ cumulants, the corresponding characteristic function has $n+1$ derivatives at $\omega = 0$, and the cumulants are related to the derivatives of the characteristic function through

$$\kappa_{k,t} = i^{-k} \frac{d^k}{d\omega^k} \log \varphi_{x,t}(\omega)\Big|_{\omega=0} \tag{A.12}$$

for any $k \leq n+1$.[2] Taking the Itô differential of the above equation (and applying the dominated convergence theorem to switch the order of $d$ and $d^k/d\omega^k$),

$$d\kappa_{k,t} = i^{-k} \frac{d^k}{d\omega^k}\left(d \log \varphi_{x,t}(\omega)\right)\Big|_{\omega=0}. \tag{A.13}$$

The remainder of the proof calculates the $k$th derivative of the right-hand side of (A.11) for $k \leq n+1$.

For the **first term**, since $x_t$ has $n$ moments, for any $\omega$ in a sufficiently small neighborhood of the origin,

$$\mathbb{E}_t[d\exp(i\omega x_t)] = \sum_{j=0}^{n+1} \frac{(i\omega)^j}{j!} \mathbb{E}_t[d(x_t^j)] + o(\omega^{n+1}). \tag{A.14}$$

[2] All the results on characteristic functions, moments, and cumulants used here can be found in Chapter 2 of Lukacs (1970).

Therefore,

$$\frac{d^k}{d\omega^k}\mathbb{E}_t[d\exp(i\omega x_t)]\bigg|_{\omega=0} = i^k\sum_{j=k}^{n+1}\frac{(i\omega)^{j-k}}{(j-k)!}\mathbb{E}_t[d(x_t^j)]\bigg|_{\omega=0} = i^k\mathbb{E}_t[d(x_t^k)]. \tag{A.15}$$

The Leibniz rule then yields

$$\frac{d^k}{d\omega^k}\frac{\mathbb{E}_t[d\exp(i\omega x_t)]}{\mathbb{E}_t[\exp(i\omega x_t)]}\bigg|_{\omega=0} = \sum_{j=0}^{k}\binom{k}{j}\left(\frac{d^{k-j}}{d\omega^{k-j}}\mathbb{E}_t[d\exp(i\omega x_t)]\bigg|_{\omega=0}\right)\left(\frac{d^j}{d\omega^j}\left(\mathbb{E}_t[\exp(i\omega x_t)]\right)^{-1}\bigg|_{\omega=0}\right). \tag{A.16}$$

Finally, note that $\left(\mathbb{E}_t[\exp(i\omega x_t)]\right)^{-1} = \exp\left(-\log\varphi_{x,t}(\omega)\right)$, and the complete exponential Bell polynomials can be used to transform the right-hand side of (A.16) into (see Comtet (1974) section 3.3)

$$\frac{d^k}{d\omega^k}\frac{\mathbb{E}_t[d\exp(i\omega x_t)]}{\mathbb{E}_t[\exp(i\omega x_t)]}\bigg|_{\omega=0} = i^k\sum_{j=0}^{k}\binom{k}{j}B_j\left(-\kappa_{1,t},\dots,-\kappa_{j,t}\right)\mathbb{E}_t[d(x_t^{k-j})] \tag{A.17}$$

$$= i^k\sum_{j=1}^{k}\binom{k}{j}B_{k-j}\left(-\kappa_{1,t},\dots,-\kappa_{k-j,t}\right)\mathbb{E}_t[d(x_t^{j})]. \tag{A.18}$$

For the **second term** on the right-hand side of (A.11), we have

$$\frac{\text{cov}_t(x_t,\exp(i\omega x_t))}{\mathbb{E}_t[\exp(i\omega x_t)]} = \frac{\mathbb{E}_t[x_t\exp(i\omega x_t)]}{\mathbb{E}_t[\exp(i\omega x_t)]} - \mathbb{E}_t[x_t] = i^{-1}\frac{d}{d\omega}\log\varphi_{x,t}(\omega) - \mathbb{E}_t[x_t]. \tag{A.19}$$

Therefore,

$$\frac{d^k}{d\omega^k}\frac{\text{cov}_t(x_t,\exp(i\omega x_t))}{\mathbb{E}_t[\exp(i\omega x_t)]}\bigg|_{\omega=0} = i^{-1}\frac{d^{k+1}}{d\omega^{k+1}}\log\varphi_{x,t}(\omega)\bigg|_{\omega=0} = i^k\kappa_{k+1,t}, \tag{A.20}$$

and the $k$th derivative of the second term in (A.11), evaluated at $\omega = 0$, is given by

$$i^k\frac{\kappa_{k+1,t}}{\sigma_{Y,t}^2}\left(dY_t - \mathbb{E}_t[x_t]dt\right). \tag{A.21}$$

Finally, for the **third term** in (A.11), the Leibniz rule combined with the results above gives

$$\frac{d^k}{d\omega^k}\left(\frac{\mathrm{cov}_t(x_t,\exp(i\omega x_t))}{\mathbb{E}_t[\exp(i\omega x_t)]}\right)^2\Bigg|_{\omega=0} \tag{A.22}$$

$$=\sum_{j=1}^{k-1}\binom{k}{j}\left(\frac{d^j}{d\omega^j}\frac{\mathrm{cov}_t(x_t,\exp(i\omega x_t))}{\mathbb{E}_t[\exp(i\omega x_t)]}\Bigg|_{\omega=0}\right)\left(\frac{d^{k-j}}{d\omega^{k-j}}\frac{\mathrm{cov}_t(x_t,\exp(i\omega x_t))}{\mathbb{E}_t[\exp(i\omega x_t)]}\Bigg|_{\omega=0}\right) \tag{A.23}$$

$$=\sum_{j=1}^{k-1}\binom{k}{j} i^j\kappa_{j+1,t} i^{k-j}\kappa_{k-j+1,t} \tag{A.24}$$

$$= i^k\sum_{j=2}^{k}\binom{k}{j-1}\kappa_{j,t}\kappa_{k-j+2,t}. \tag{A.25}$$

Putting everything together and canceling the $i^k$ constants completes the proof of the theorem.

## A.2 Proof of proposition 1

The dynamics of the first moment are exactly as in theorem 1. For the second moment, we simply need to show that

$$\sum_{j=1}^{2}\binom{2}{j} B_{2-j}\left(-\kappa_{1,t},\ldots,-\kappa_{2-j,t}\right)\mathbb{E}_t[d(x_t^j)] = \mathbb{E}_t\left[d\left\langle x\right\rangle_t\right] + 2\,\mathrm{cov}_t\left(x_t, dx_t\right). \tag{A.26}$$

The left-hand side of the above expression is given by

$$\mathbb{E}_t[d(x_t^2)] - 2\mathbb{E}_t[x_t]\mathbb{E}_t[dx_t]. \tag{A.27}$$

Assumption 3 implies that $x_t$ is a semimartingale of the form $x_t = x_0 + M_t + A_t$, where $M_t$ is a right-continuous martingale and $A_t$ is an absolutely continuous process. By Itô's lemma for semimartingales (e.g., Theorem 32 of Protter (2005)),

$$\mathbb{E}_t[d(x_t^2)] = 2\mathbb{E}_t[x_{t^-}dx_t] + \mathbb{E}_t\left[d\left\langle x\right\rangle_t\right], \tag{A.28}$$

where $x_{t^-}\equiv\lim_{s\uparrow t}x_s$. By assumption 3, $x_t = x_{t^-} + \Delta M_t$, where $\Delta M_t$ denotes the time-$t$ jump of $M$. Since $Y_t$ is a continuous process, its natural filtration is continuous. Therefore,

$$\mathbb{E}_t[\Delta M_t] = \mathbb{E}_t[\Delta M_t|Y^t] = \mathbb{E}_t[\Delta M_t|Y^{t^-}] = 0, \tag{A.29}$$

where the last equality is a consequence of the fact that $M$ is a martingale. Thus, $\mathbb{E}_t[x_t] = \mathbb{E}_t[x_{t-}]$, and so

$$\mathbb{E}_t[d(x_t^2)] - 2\mathbb{E}_t[x_t]\mathbb{E}_t[dx_t] = \mathbb{E}_t\left[d\left\langle x\right\rangle_t\right] + 2\,\mathrm{cov}_t\left(x_t, dx_t\right), \tag{A.30}$$

where $\mathrm{cov}_t\left(x_t, dx_t\right) \equiv \mathbb{E}_t[x_{t-}dx_t] - \mathbb{E}_t[x_{t-}]\mathbb{E}_t[dx_t]$.

## A.3 Additional derivations

### A.3.1 Cash-flows as the signal

While the paper in general leaves the information stream unstructured, one interpretation is that it represents cash-flow realizations. That is the setup used in Veronesi (1999), for example. The paper's assumption for $Y$ can accommodate that in certain cases.

First, as a slight generalization, denote the cumulative cash-flow by $\mathcal{D}_t$, and assume a constant discount rate so that

$$P_t\left(\mathcal{I}_t\right) = \mathbb{E}\left[\int_{s=0}^{\infty} \exp\left(-\rho s\right) d\mathcal{D}_{t+s} \mid \mathcal{I}_t\right] \tag{A.31}$$

If $\mathcal{D}$ satisfies

$$d\mathcal{D}_t = z_t dt + \sigma_{D,t} dB_t \tag{A.32}$$

where $B_t$ is a Brownian motion with respect to $\theta_t$ and $z_t$ is an arbitrary martingale, then

$$X_t = \rho^{-1} z_t \tag{A.33}$$

$$\text{and } d\mathcal{D}_t = \rho X_t dt + \sigma_{D,t} dB_t \tag{A.34}$$

That is, if cash-flows are equal to the continuous-time analog to a random walk plus noise, then cash flows themselves can be taken to be the signal about the fundamental value – the random walk component.

A second and somewhat different potential model is that perhaps firms have complete information – they can see $\theta_t$ – and they set cash-flows to equal the NPV – i.e. $P_t\left(\theta_t\right)$ – plus a noise component that might come from their own liquidity demand or other shocks that affect their ability or desire to make dividend payments. Then, by construction, cash flows are exactly equal to $X_t dt$ plus noise.

That said, both of these examples are obviously contrived, and in practice cash flows are really only one signal among many that investors receive.

### A.3.2 Equation (16)

Doing a second order approximation of the price process using proposition 1,

$$\mathbb{E}_t[(p_{t+\Delta t} - \mathbb{E}_t[p_{t+\Delta t}])^2] = \kappa_{2,t}^2 \sigma_{Y,t}^{-2} \Delta t + O(\Delta t^2), \tag{A.35}$$

and

$$\mathbb{E}_t[(p_{t+\Delta t} - \mathbb{E}_t[p_{t+\Delta t}])^3] = \mathbb{E}_t\left[\left(\kappa_{2,t}\sigma_{Y,t}^{-1}\Delta W_t + \frac{\kappa_{3,t}\sigma_{Y,t}^{-1}}{2}(\Delta W_t^2 - \Delta t)\right)^3\right] + O(\Delta t^{5/2}) \tag{A.36}$$

$$= 3\kappa_{2,t}^2 \kappa_{3,t} \sigma_{Y,t}^{-4} \Delta t^2 + O(\Delta t^{5/2}). \tag{A.37}$$

Therefore,

$$skew_t\,(p_{t+\Delta t})\,(\Delta t)^{-1/2} = \frac{\mathbb{E}_t[(p_{t+\Delta t} - \mathbb{E}_t[p_{t+\Delta t}])^3]\,(\Delta t)^{-1/2}}{\left(\mathbb{E}_t[(p_{t+\Delta t} - \mathbb{E}_t[p_{t+\Delta t}])^2]\right)^{3/2}} \tag{A.38}$$

$$= \frac{\kappa_{3,t}}{\kappa_{2,t}} \sigma_{Y,t}^{-1} + O(\Delta t^{1/2}). \tag{A.39}$$

### A.3.3 Equation (24)

Starting from equation (8),

$$d\left(\frac{\kappa_{2,t}}{\sigma_{Y,t}}\right) = \frac{1}{\sigma_{Y,t}} \frac{\kappa_{3,t}}{\sigma_{Y,t}^2} \left(dY_t - \mathbb{E}_t\left[x_t\right] dt\right) + \mathbb{E}_t\left[d\left(x_t^2\right)\right] - \frac{1}{\sigma_{Y,t}} \left(\frac{\kappa_{2,t}}{\sigma_{Y,t}}\right)^2 dt \tag{A.40}$$

$$= \frac{1}{\sigma_{Y,t}} \frac{\kappa_{3,t}}{\kappa_{2,t}} \frac{\kappa_{2,t}}{\sigma_{Y,t}^2} \left(dY_t - \mathbb{E}_t\left[x_t\right] dt\right) + \mathbb{E}_t\left[d\left(x_t^2\right)\right] - \frac{1}{\sigma_{Y,t}} \left(\frac{\kappa_{2,t}}{\sigma_{Y,t}}\right)^2 dt \tag{A.41}$$

$$= \frac{1}{\sigma_{Y,t}^2} \left(skew_{t\to t+\Delta t}\left(dp_t\right)\left(\Delta t\right)^{-1/2}\right) \frac{1}{3} \frac{\kappa_{2,t}}{\sigma_{Y,t}^2} \left(dY_t - \mathbb{E}_t\left[x_t\right] dt\right) + \mathbb{E}_t\left[d\left(x_t^2\right)\right] - \frac{1}{\sigma_{Y,t}} \left(\frac{\kappa_{2,t}}{\sigma_{Y,t}}\right)^2 dt \tag{A.42}$$

$$= \frac{1}{\sigma_{Y,t}^2} \left(skew_{t\to t+\Delta t}\left(dp_t\right)\left(\Delta t\right)^{-1/2}\right) \frac{1}{3} \left[dp_t - \mathbb{E}_t dx_t\right] + \mathbb{E}_t\left[d\left(x_t^2\right)\right] - \frac{1}{\sigma_{Y,t}} \left(\frac{\kappa_{2,t}}{\sigma_{Y,t}}\right)^2 dt, \tag{A.43}$$

where the third line uses equation (16) and the fourth line inserts the formula for $dp_t = d\kappa_{1,t}$.

### A.3.4 Solving the filtering problem in logs

This section analyzes the filtering problem without applying the approximation in equation (4) (and we note here again that the simulation in section 6 does not use equation (4) either). Specifically, instead of assuming $p_t = \mathbb{E}_t[x_t]$, we set $p_t \equiv \log \mathbb{E}_t[\exp(x_t)]$. Then we obtain the following counterpart to corollary 2.

**Proposition 5** *When $\sigma_{Y,t}$ is constant, $vol_t$ follows a diffusion satisfying*

$$\begin{aligned} d\left(vol_t\right) = & \frac{1}{\sigma_{Y,t}} \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{j=1}^{k} \binom{k}{j} B_{k-j}\left(-\kappa_{1,t}, \ldots, -\kappa_{k-j,t}\right) \mathbb{E}_t[d(x_t^j)] \\ & + \frac{1}{\sigma_{Y,t}^3} \sum_{k=2}^{\infty} \frac{\kappa_{k+1,t}}{(k-1)!} \left(dY_t - \mathbb{E}_t[x_t]dt\right) \\ & - \frac{1}{2\sigma_{Y,t}^3} \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{j=2}^{k} \binom{k}{j-1} \kappa_{j,t}\kappa_{k-j+2,t} dt. \end{aligned} \tag{A.44}$$

In this case, instead of just depending on the third moment, the sign of the leverage effect is more complicated. In particular, the analog to proposition 2 is

$$\frac{\operatorname{cov}\left(dp_t, dvol_t\right)}{\operatorname{var}\left(dp_t\right)} = \frac{\sum_{k=2}^{\infty} \frac{\kappa_{k+1,t}}{(k-1)!}}{\sum_{k=1}^{\infty} \frac{\kappa_{k+1,t}}{k!}} \sigma_{Y,t}^{-1} \tag{A.45}$$

and the equation in proposition 3 then becomes

$$\lim_{\Delta t \downarrow 0} skew_t\left(p_{t+\Delta t}\right)\left(\Delta t\right)^{-1/2} = 3\frac{\sum_{k=2}^{\infty} \frac{\kappa_{k+1,t}}{(k-1)!}}{\sum_{k=1}^{\infty} \frac{\kappa_{k+1,t}}{k!}} \sigma_{Y,t}^{-1} \tag{A.46}$$

What is in the main text has only the leading terms from the two power series. Note also that these together immediately imply that corollary 3 holds without any changes.

In moving on to the empirical analysis, we have the following modification.

**Corollary 5** *The analog to equation (24) is*

$$\begin{aligned} d\left(vol_t\right) = & \left(\frac{1}{3}\Delta t^{-1/2}\right) skew_t\left(p_{t+\Delta t}\right)\left(dp_t - \mathbb{E}_t[dp_t]\right) \\ & - vol_t^2 \left(\frac{\kappa_{2,t}}{\sigma_{Y,t}^2 vol_t} + \left(\frac{1}{3}\Delta t^{-1/2}\right) skew_t\left(p_{t+\Delta t}\right)\right) dt \\ & + \frac{1}{\sigma_{Y,t}} \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{j=1}^{k} \binom{k}{j} B_{k-j}\left(-\kappa_{1,t}, \ldots, -\kappa_{k-j,t}\right) \mathbb{E}_t[d(x_t^j)]. \end{aligned} \tag{A.47}$$

The first term is identical to the first term in regression (24). In other words, the $p_t = \mathbb{E}[x_t \mid Y^t]$ approximation has no bearing for the relationship between the leverage effect and return skewness. The second and third terms are different than those in (24). However, assuming that $\kappa_{k,t}$ is small for all $k \geq 4$, the price is a martingale, and $x_t$ has stationary and independent increments, we have

$$d(vol_t) \approx \left(\frac{1}{3}\Delta t^{-1/2}\right) skew_t(p_{t+\Delta t})\, dp_t - \frac{1}{\sigma_{Y,t}}\left(1 + \sigma_{Y,t}\left(\frac{1}{6}\Delta t^{-1/2}\right) skew_t(p_{t+\Delta t})\right) vol_t^2 dt + \text{constant} \cdot dt, \tag{A.48}$$

which is identical to (24), except for the $\sigma_{Y,t}\left(\frac{1}{6}\Delta t^{-1/2}\right) skew_t(p_{t+\Delta t})$ correction in the coefficient of $vol_t^2$. The $p_t = \mathbb{E}_t[x_t]$ approximation introduces a Jensen's inequality error term that has an expansion in the cumulants; $\sigma_{Y,t}\left(\frac{1}{6}\Delta t^{-1/2}\right) skew_t(p_{t+\Delta t})$ is the leading term in that expansion.

### A.3.5 Accommodating the information structure in an equilibrium model

This section has two parts. First, it shows how to derive equations (4) and (5) in a Lucas tree economy in which agents price assets based on a single signal, $Y$. The basic setup in equation (1) holds in such models, but the restriction to a single signal in the completely general case is less obvious. Second, it gives a description of how the information structure assumed in section 3 can be incorporated into more general models.

#### A.3.5.1 Lucas tree economy

There is a single tree with a cash-flow of $D_t$. Agents are all identical. They are endowed with a unit claim on the tree, which pays $D_t$ in each period. Their date-$t$ budget constraint is

$$\mathbb{E}_t \int_{j=0}^{\infty} M_{t+j} C_{t+j} dj = M_t P_t + \mathbb{E}_t \int_{j=0}^{\infty} M_{t+j} D_{t+j} dj \tag{A.49}$$

where $M_{t+j}$ is the price of a date-$t+j$ Arrow–Debreu security.

The agents' objective is

$$\max \mathbb{E}_t \left[\int_{j=0}^{\infty} \beta^j u(C_{t+j})\, dj \mid Y^t\right] \tag{A.50}$$

where $u$ represents utility over consumption, $C$, and $Y^t$ is the history of the signal process up to date $t$. We assume that agents' trading decision on date $t$, as represented by their holdings of claims on the tree, must be measurable with respect to $Y^t$. Their consumption is then a residual.

Consider a perturbation at the optimum that purchases one additional (infinitesimal) unit of the tree on date $t$ – raising consumption in proportion to $D_{t+j}$ on all future dates, and reducing consumption in proportion to $P_t$ on date $t$. At the optimum, it must be the case that

$$\mathbb{E}\left[P_t u'(C_t) \mid Y^t\right] = \mathbb{E}\left[\int_{j=0}^{\infty} u'(C_{t+j}) D_{t+j} dj \mid Y^t\right] \tag{A.51}$$

The interpretation of the right-hand side is standard. The left-hand side is more subtle. It says that the cost of the purchase of an additional unit of the tree is equal to the expected marginal utility of consumption conditional on the value of $Y^t$, since it will reduce consumption by $P_t$ in all states of the world in which $Y^t$ takes on that particular value.

Rearranging and noting that $P_t$ can only be a function of $Y^t$,

$$P_t = \frac{\mathbb{E}\left[\int_{j=0}^{\infty} u'(C_{t+j}) D_{t+j} dj \mid Y^t\right]}{\mathbb{E}\left[u'(C_t) \mid Y^t\right]} \tag{A.52}$$

**As a first observation** note that if cash-flows are pre-determined and thus measurable with respect to $Y^t$, then $\mathbb{E}\left[u'(C_t) \mid Y^t\right] = u'(C_t)$ and equation (A.52) reduces to equation (1) with $M_t \equiv u'(C_t)$ and hence (4) and (5) follow immediately.

Alternatively, suppose cash-flows are not predetermined. Then consider equation (5) with $M_t \equiv u'(C_t)$:

$$\begin{aligned} x_t &= \mathbb{E}\left[\log \int_{s=0}^{\infty} \frac{D_{t+s} M_{t+s}}{M_t} ds \mid \theta_t\right] && \text{(A.53)} \\ &= \mathbb{E}\left[\log \int_{s=0}^{\infty} D_{t+s} u'(C_{t+s}) ds \mid \theta_t\right] - \mathbb{E}\left[\log u'(C_t) \mid \theta_t\right] && \text{(A.54)} \end{aligned}$$

Taking the log of (A.52),

$$\log P_t = \log \mathbb{E}\left[\int_{j=0}^{\infty} u'(C_{t+j}) D_{t+j} dj \mid Y^t\right] - \log \mathbb{E}\left[u'(C_t) \mid Y^t\right] \tag{A.55}$$

As in the main text, passing the log through the expectation yields

$$\begin{aligned} p_t &= \mathbb{E}\left[\log \int_{j=0}^{\infty} u'(C_{t+j}) D_{t+j} dj \mid Y^t\right] - \mathbb{E}\left[\log u'(C_t) \mid Y^t\right] && \text{(A.56)} \\ &= \mathbb{E}\left[x_t \mid Y^t\right] && \text{(A.57)} \end{aligned}$$

#### A.3.5.2 General setup

For a general model, we retain equation (1),

$$P_t = \mathbb{E}\left[\int_0^\infty \frac{M_{t+s}D_{t+s}ds}{M_t} \mid \mathcal{F}_t\right] \tag{A.58}$$

where $\mathcal{F}$ here represents the natural filtration induced by $Y^t$ (i.e. $\mathcal{F}_t$ is the sigma-field induced by $Y^t$). Instead of the $\theta$ notation, for consistency here, denote a second filtration, $\mathcal{G}_t$, such that $\mathcal{F}_t$ is coarser than $\mathcal{G}_t$. Then define

$$X_t \equiv \mathbb{E}\left[\int_0^\infty \frac{M_{t+s}D_{t+s}ds}{M_t} \mid \mathcal{G}_t\right] \tag{A.59}$$

$$dY_t = X_t dt + \sigma_{Y,t} dW_t \tag{A.60}$$

and set $M_t$ to be the equilibrium state price process given the filtration $\mathcal{F}_t$. Note that in general models state prices may depend on information, either via endogenous consumption decisions or because information itself affects marginal utility (as in recursive preferences, for example).

We then have, by the law of iterated expectations and the assumption that $\mathcal{F}_t$ is coarser than $\mathcal{G}_t$,

$$P_t = \mathbb{E}\left[X_t \mid \mathcal{F}_t\right] \tag{A.61}$$

as in the main text. The equations, however, define a fixed point – $\mathcal{F}$ depends on $Y$, which depends on $X$, which depends on $M$, which depends on $\mathcal{F}$.

Such a fixed point obviously need not necessarily exist or be unique in any given setting. The analysis in the paper is not meant to fully specify the model but rather to essentially study properties of any model that happens to satisfy (A.61) where $\mathcal{F}$ is the filtration induced by the $Y$ in (A.60).

### A.3.6 The effects of signals being priced

This section relaxes the assumption from the main analysis that the noise in the signals is not priced. It examines a simple, if somewhat informal, extension of the baseline model. Specifically, consider a case where the only priced risk is $dW_t$, so that under full information,

$$X_t = E\left[\int_0^\infty \exp\left(-r_f s\right) D_{t+s}ds \mid \theta_t\right] \tag{A.62}$$

Whereas in the baseline case we have $p_t = E[x_t \mid Y^t]$ here we allow for discount rates to enter via

$$p_t = E\left[x_t \mid y^t\right] - \gamma \int_0^\infty \exp(-r_f s)\, d\langle p \rangle_{t+s} \tag{A.63}$$

Equation (A.63) takes the form of the Campbell–Shiller approximation. As Campbell (2017) discusses, the Campbell–Shiller approximation does not have a direct analog in continuous time, so this equation is just heuristic. More formal approaches are to study an arithmetic model (Veronesi (1999)), assume specific functional forms (e.g. Ang and Liu (2007)), analyze zero-coupon dividend claims (Lettau and Wachter (2007)), or simply analyze the model numerically. A fully formal derivation is beyond the scope of this paper (and would require breaking new ground, solving equations that are not known to have analytic solutions).

The core point is that equation (A.63) allows us to examine the effects of time-varying endogenous discount rates on prices (as opposed to those encoded in an exogenous SDF, $M$, which the main analysis allows). As the simplest possible case, equation is treating risk premia as coming from a myopic CAPM type specification for expected returns, where expected returns depend on their local volatility, $d\langle p \rangle_{t+s}$. In that sense there is a constant risk price, $\gamma$, on shocks to prices.

Equation (A.63) defines a fixed-point problem, which represents the fundamental difficulty in solving these models in closed form. Veronesi (1999), for example, solves the fixed point problem numerically. To develop an understanding of the model, though, we consider an iterative approach. Define

$$p_{0,t} \equiv E\left[x_t \mid Y^t\right] \tag{A.64}$$

$p_{0,t}$ represents the price process when discount rates are not endogenous to the volatility of prices. The natural first step is to then incorporate risk premia based on the conditional volatility of $p_{0,t}$,

$$\begin{aligned} p_{1,t} &\equiv E_t[x_t] - \gamma E_t\left[\int_0^\infty \exp(-r_f s)\, d\langle p_0 \rangle_{t+s}\right] && \text{(A.65)} \\ &= E_t[x_t] - \gamma \int_0^\infty \exp(-r_f s)\, E_t\left[\kappa_{2,t+s}^2\right] \sigma_y^{-2} ds && \text{(A.66)} \end{aligned}$$

(where we take $\sigma_y$ as constant here for simplicity).

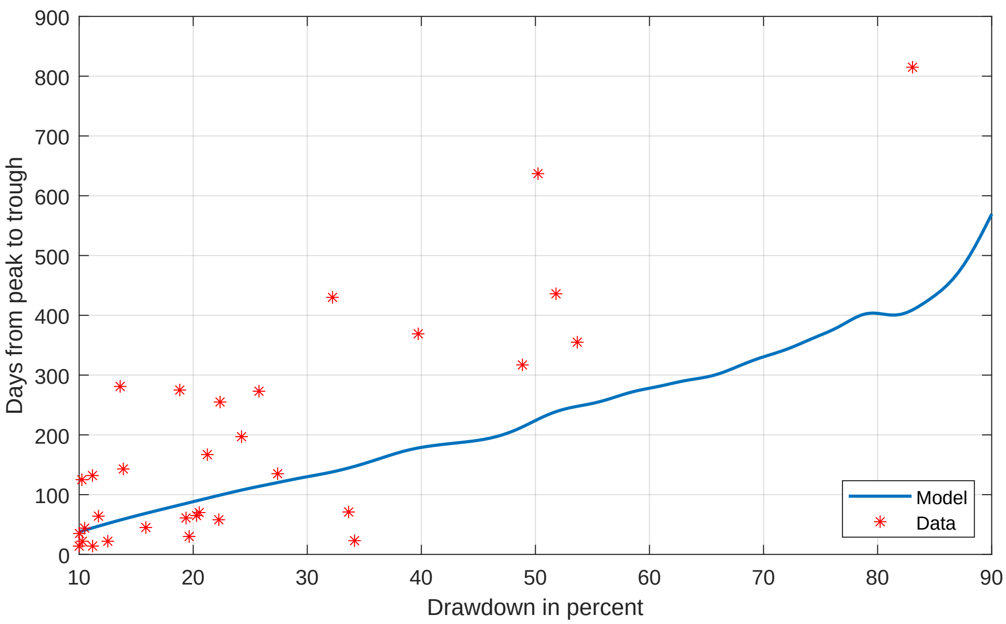
The return innovation in each period is then equal to the innovation in $E_t[x_t]$, representing cash flow news, minus the innovation in $\gamma \int_0^\infty \exp(-r_f s)\, E_t\left[\kappa_{2,t+s}^2\right] \sigma_y^{-2} ds$, representing discount rate news.

Both terms are measurable with respect to $dW_t$. The loading of the first term on $dW_t$ remains $\kappa_{2,t}\sigma_y^{-2}$. The loading on the second term depends on how expectations of

$\kappa^2_{2,t+s}$ respond to signals. In principle, that expectation depends on all of the higher-order cumulants. However, the direct effect of a shock $dW_t$ on $\kappa_{2,t}$ is equal to $\kappa_{3,t}\sigma_y^{-2}$. That strongly suggests (though may not imply in all settings) that when $\kappa_{3,t} > 0$, positive realizations of $dW_t$ increase volatility and hence discount rates, driving prices down, while when $\kappa_{3,t} < 0$ – the case the paper argues is empirically relevant – positive realizations of $dW_t$ reduce volatility and discount rates, driving prices up.

The analysis therefore suggests that when $\kappa_{3,t} < 0$, cash-flow and discount-rate news push in the same direction, and time-varying discount rates therefore raise the volatility of prices. The model would additionally generate countercyclical risk premia, with expected returns rising following bad news. That having been said, it is well known that it is very difficult to find any positive relationship between the expected mean and variance of US stock market returns.

Figure A.1: Time from peak to trough in large drawdowns




**Note:** The blue line plots the average number of days from peak to trough in the model simulations for a drawdown (decline relative to the running maximum) of a given size. The average is calculated using a kernel smoother on the simulated realized drawdowns. The red stars represent the time from peak to trough for drawdowns observed in the US stock market.
Because the behavior of drawdowns depends not only on the equity premium but also the risk-free rate, we use the following procedure to put the data and simulation on the same footing. For the data, drawdowns are calculated based on cumulative adjusted returns, where the adjusted return is defined as the excess return plus 3.3% per year, which is the average risk-free rate in the sample. In the model, we set the mean growth rate of prices to be equal to 10.9%, which corresponds to the mean of the equity premium plus the mean risk-free rate (i.e. the mean total return on equities, corresponding to $g$ in the model). That ensures that both series have the same average growth rate which makes their drawdowns comparable.