Appendix to: "Intangible Capital, Relative Asset Shortages and Bubbles"

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Proofs

Proof of Proposition 1

It is clear that $K_t(t+1) = 0$, $Z_t(t+1) = 0$ for all t satisfy the equilibrium conditions reported in Section 4.1, for a certain value of $\lambda_t \geq 0$.

First, we prove that if a nontrivial bubbleless equilibrium exists in the parameter region B, the constraint has to be binding at all times. Suppose that $(\alpha, \beta, \pi) \in B$ but the constraint is not binding for at least one generation t. Then, $\lambda_t = 0$, which implies:

$$k_t(t+1) = \frac{\alpha}{\beta} z_t(t+1)$$

The following inequality must hold:

$$z_t(t+1) \leq w(t+1)$$

or

$$Z_t(t+1) \leq \pi w(t+1)$$

Equilibrium in the savings market implies:

$$K_t(t+1) + Z_t(t+1) = w(t+1)$$

so:

$$Z_t(t+1) \leq \pi \left(K_t(t+1) + Z_t(t+1)\right)$$

or:

$$(1-\pi)\beta \leq \pi\alpha$$

which coincides with $NB = \Theta \backslash B$.

Conversely, suppose that an equilibrium exists for the parameter region NB, but the constraint is binding for at least one period t. Then, $\lambda_t > 0$ and we must have:

$$Z_t(t+1) = \pi w(t+1)$$

so that

$$Z_t(t+1) = \frac{\pi}{(1-\pi)} K_t(t+1)$$

Besides, remember that

$$\lambda_t = IRR(t+2) - R(t+2) = \beta \frac{Y_t(t+2)}{Z_t(t+1)} - \alpha \frac{Y_t(t+2)}{K_t(t+1)} = Y_t(t+2)\left(\frac{\beta}{Z_t(t+1)} - \frac{\alpha}{K_t(t+1)}\right)$$

Since in a non-trivial equilibrium $Y_t(t+2) > 0$, we must have:

$$\frac{\beta}{Z_t(t+1)} - \frac{\alpha}{K_t(t+1)} > 0$$

or

$$\beta K_t(t+1) > \alpha Z_t(t+1) = \alpha \frac{\pi}{(1-\pi)} K_t(t+1)$$

Therefore, if the constraint is binding in period t we must have:

$$(1-\pi)\beta > \pi\alpha$$

which coincides with the B region.

We have then shown that if an equilibrium exists without bubbles it must either have the constraint binding at all times (region B), or never binding (region NB). We will now prove the existence of the dynamic equilibrium separately in each region, by construction.

Let us start with the equilibrium in the B region. As reported in the text, the equilibrium must satisfy the following conditions:

$$w(t+1) = (1-\alpha-\beta) Y_{t-1}(t+1)$$

$$R(t+2) = \alpha \frac{Y_t(t+2)}{K_t(t+1)}$$

$$K_t(t+1) + Z_t(t+1) = w(t+1) = (1-\alpha-\beta)\, Y_{t-1}(t+1)$$

$$Z_t(t+1) \leq \pi w(t+1)$$

$$IRR(t+2) = R(t+2) + \lambda_t = \frac{\beta Y_t(t+2)}{Z_t(t+1)}$$

$$\lambda_t \left[Z_t(t+1) - \pi w(t+1)\right] = 0$$

$$\lambda_t \geq 0$$

$$Y_t(t+2) = A(t+2)K_t(t+1)^{\alpha} Z_t(t+1)^{\beta}$$

An equilibrium exists if the system above admits a solution for all t.

In the B region, for every t and given $Y_{t-1}(t+1)$, the vector $[Y_t(t+2), K_t(t+1), Z_t(t+1)]$ is determined by the solution to the system of equations:

$$K_t(t+1) + Z_t(t+1) = (1 - \alpha - \beta)\, Y_{t-1}(t+1)$$

$$Z_t(t+1) = \frac{\pi}{(1-\pi)} K_t(t+1)$$

$$Y_t(t+2) = A(t+2)K_t(t+1)^{\alpha} Z_t(t+1)^{\beta}$$

and prices can then be backed out through the optimality conditions of the firm. The system can be further simplified to:

$$K_t(t+1) = (1-\pi)\,(1-\alpha-\beta)\, Y_{t-1}(t+1)$$

$$Y_t(t+2) = A(t+2)K_t(t+1)^{\alpha} \left(\frac{\pi}{(1-\pi)} K_t(t+1)\right)^{\beta}$$

and finally

$$Y_t(t+2) = A(t+2)\left(\frac{\pi}{(1-\pi)}\right)^{\beta} \left((1-\pi)\,(1-\alpha-\beta)\, Y_{t-1}(t+1)\right)^{\alpha+\beta}$$

This shows the existence of the equilibrium. For every $Y_{t-1}(t+1) > 0$, this equation tells us how $Y_t(t+2)$ is determined along the equilibrium path, and through the equations reported above we can always find $Z_t(t+1) > 0$, $K_t(t+1) > 0$ and then prices that satisfy all the equilibrium conditions.

Finally, we can rescale the system by $A(t+2)^{\frac{1}{1-\alpha-\beta}}$ in order to obtain the steady state. Denote

$$\hat{Y}_t(t+2) = \frac{Y_t(t+2)}{A(t+2)^{\frac{1}{1-\alpha-\beta}}}$$

and note that:

$$\frac{Y_{t-1}(t+1)}{A(t+2)^{\frac{1}{1-\alpha-\beta}}} = \frac{Y_{t-1}(t+1)}{(A(t+1)(1+n))^{\frac{1}{1-\alpha-\beta}}} = \frac{\hat{Y}_{t-1}(t+1)}{1+g}$$

In the B region, the system becomes:

$$\hat{Y}_t(t+2) = \left(\frac{\pi}{(1-\pi)}\right)^{\beta} \left((1-\pi)(1-\alpha-\beta)\hat{Y}_{t-1}(t+1)\right)^{\alpha+\beta} (1+g)^{-(\alpha+\beta)}$$

Steady state capital and intangibles satisfy:

$$\hat{K}_t(t+1) + \hat{Z}_t(t+1) = (1-\alpha-\beta)\frac{\hat{Y}_{t-1}(t+1)}{(1+g)}$$

$$\hat{Z}_t(t+1) = \frac{\pi}{(1-\pi)}\hat{K}_t(t+1)$$

Finally, the rates of return satisfy:

$$R(t+2) = \alpha\frac{\hat{Y}_t(t+2)}{\hat{K}_t(t+1)}$$

$$IRR(t+2) = \beta\frac{\hat{Y}_t(t+2)}{\hat{Z}_t(t+1)}$$

In the steady state of this economy $\hat{Y}, \hat{K}, \hat{Z}$ are constant, so the respective level variables grow at rate $1+g$. The unique steady-state is given by:

$$\hat{Y} = \left[\left(\frac{\pi}{(1-\pi)}\right)^{\beta} ((1-\pi)(1-\alpha-\beta))^{\alpha+\beta} (1+g)^{-(\alpha+\beta)}\right]^{\frac{1}{1-\alpha-\beta}}$$

$$\hat{K} = \frac{(1-\pi)(1-\alpha-\beta)}{(1+g)}\hat{Y}$$

$$\hat{Z} = \frac{\pi(1-\alpha-\beta)}{(1+g)}\hat{Y}$$

Also the interest rate R and the IRR will be constant in the long run.

$$R = \frac{\alpha(1+g)}{(1-\pi)(1-\alpha-\beta)}$$

$$IRR = \frac{\beta(1+g)}{\pi(1-\alpha-\beta)}$$

Let us now analyze the NB region. In the NB region, the equilibrium conditions simplify to:

$$w(t+1) = (1-\alpha-\beta)Y_{t-1}(t+1)$$

$$R(t+2) = \alpha\frac{Y_t(t+2)}{K_t(t+1)}$$

$$K_t(t+1) + Z_t(t+1) = (1-\alpha-\beta)\,Y_{t-1}(t+1)$$

$$IRR(t+2) = R(t+2) = \beta\frac{Y_t(t+2)}{Z_t(t+1)}$$

$$Y_t(t+2) = A(t+2)K_t(t+1)^{\alpha}Z_t(t+1)^{\beta}$$

Again, an equilibrium exists if the system above admits a solution for all t. For every t given $Y_{t-1}(t+1)$, the vector $[Y_t(t+2), K_t(t+1), Z_t(t+1)]$ is determined by the solution to the system:

$$K_t(t+1) + Z_t(t+1) = (1-\alpha-\beta)\,Y_{t-1}(t+1)$$

$$Z_t(t+1) = \frac{\beta}{\alpha} K_t(t+1)$$

$$Y_t(t+2) = A(t+2) K_t(t+1)^{\alpha} Z_t(t+1)^{\beta}$$

and prices can then be backed out from first order conditions. The system can be further simplified to:

$$K_t(t+1) = \frac{\alpha}{\alpha+\beta} (1-\alpha-\beta) Y_{t-1}(t+1)$$

$$Y_t(t+2) = A(t+2) K_t(t+1)^{\alpha} \left(\frac{\beta}{\alpha} K_t(t+1) \right)^{\beta}$$

Just like in the B case, we can represent the evolution of the system with one state variable, $Y_{t-1}(t+1)$. The equilibrium is characterized by:

$$\hat{Y}_t\left(t+2\right) = \left(\frac{\beta}{\alpha}\right)^{\beta} \left[\frac{\alpha}{\alpha+\beta} \frac{(1-\alpha-\beta)}{(1+g)} \hat{Y}_{t-1}\left(t+1\right) \right]^{\alpha+\beta}$$

$$\hat{K}_t(t+1) = \frac{\alpha}{\alpha+\beta} (1-\alpha-\beta) \frac{\hat{Y}_{t-1}(t+1)}{(1+g)}$$

$$\hat{Z}_t(t+1) = \frac{\beta}{\alpha+\beta} (1-\alpha-\beta) \frac{\hat{Y}_{t-1}(t+1)}{(1+g)}$$

In the steady-state of this economy, $\hat{Y}, \hat{K}$ and $\hat{Z}$ are constant, so the respective level variables grow at rate $1+g$. The unique steady-state is given by:

$$\hat{Y} = \left(\frac{\beta}{\alpha}\right)^{\frac{\beta}{1-\alpha-\beta}} \left[\frac{\alpha}{\alpha+\beta} \frac{(1-\alpha-\beta)}{(1+g)} \right]^{\frac{\alpha+\beta}{1-\alpha-\beta}}$$

$$\hat{K} = \frac{\alpha}{\alpha+\beta} (1-\alpha-\beta) \frac{\hat{Y}}{(1+g)}$$

$$\hat{Z} = \frac{\beta}{\alpha + \beta} (1 - \alpha - \beta) \frac{\hat{Y}}{(1 + g)}$$

$$R = IRR = \frac{(\alpha + \beta)(1 + g)}{(1 - \alpha - \beta)}$$

Proof of Lemma 2

We look first at bubbly equilibria in which the generation t entrepreneurs are financially constrained (i.e. we look for a solution of the system where $\lambda_t > 0$). In these equilibria, we must have:

$$K_t(t + 1) + Z_t(t + 1) + B (t + 1) = (1 - \alpha - \beta) Y_{t-1} (t + 1)$$

$$B (t + 1) = R (t + 1) B (t)$$

$$Z_t(t + 1) = \pi(1 - \alpha - \beta)Y_{t-1}(t + 1)$$

$$Y_t(t + 2) = A(t + 2)K_t(t + 1)^{\alpha} Z_t(t + 1)^{\beta}$$

$$R(t + 2) = \alpha \frac{Y_t(t + 2)}{K_t(t + 1)}$$

Note that given the vector $[Y_{t-1}(t+1), B(t+1)]$, the state of the system when the financing constraint is binding is determined as follows. First, $Z_t(t+1)$ is determined given $Y_{t-1}(t+1)$

$$Z_t(t + 1) = \pi(1 - \alpha - \beta)Y_{t-1}(t + 1)$$

Then, K_t is determined by the equation

$$K_t(t+1) = (1-\pi)(1-\alpha-\beta) Y_{t-1}(t+1) - B(t+1)$$

Knowing $K_t(t+1)$ and $Z_t(t+1)$ allows us to find $Y_t(t+2)$ and $B(t+2)$:

$$Y_t(t+2) = A(t+2) K_t(t+1)^{\alpha} Z_t(t+1)^{\beta}$$

and

$$B(t+2) = \alpha \frac{Y_t(t+2)}{K_t(t+1)} B(t+1)$$

These equations show two things. First, that $[Y_{t-1}(t+1), B(t+1)]$ plus the knowledge that $\lambda_t > 0$ is enough to determine the state of the economy in the next period. Second, they show that a necessary condition for the existence of a binding equilibrium in which, after $[Y_{t-1}(t+1), B(t+1)]$, the financing constraint is binding is that

$$K_t(t+1) = (1-\pi)(1-\alpha-\beta) Y_{t-1}(t+1) - B(t+1) \geq 0$$

or

$$B(t+1) \leq (1-\pi)(1-\alpha-\beta) Y_{t-1}(t+1)$$

Note that this is a condition that depends on the state of the economy $[Y_{t-1}(t+1), B(t+1)]$. It is saying that if the economy at time $t+1$ has too big a bubble relative to output, in the next generation there cannot exist a continuation of the equilibrium in which the financing constraint is binding.

The other condition that needs to hold for the system to show a binding constraint is that $\lambda_t > 0$, or

$$\beta K_t(t+1) > \alpha Z_t(t+1)$$

In turn, this implies

$$B\left(t+1\right) < [(1-\pi) - \frac{\alpha}{\beta}\pi](1-\alpha-\beta)Y_{t-1}(t+1)$$

Hence this is another necessary condition that constrains the maximum value of the bubble. This constraint can be satisfied only if $\beta(1-\pi) - \pi\alpha > 0$, which coincides with region B.

This proves that in the parameter region NB no equilibrium path can support a binding financing constraint. Since the term in square brackets is also always less than 1, this constraint is always tighter than the one above, so it always dominates it.

Hence, an equilibrium which supports the path $[Y_{t-1}(t+1), B(t+1)]$ has the following features:

i) In the region NB, the financing constraint can never be binding along any equilibrium path. Because of this, $[Y_{t-1}(t+1), B(t+1)]$ is a state vector in this region of the parameter space.

ii) If the financing constraint is binding for generation t, then it must be that

$$B\left(t+1\right) < [(1-\pi) - \frac{\alpha}{\beta}\pi](1-\alpha-\beta)Y_{t-1}(t+1) = B_{max,b}(Y_{t-1}(t+1))$$

This condition is also sufficient for the existence of the intratemporal equilibrium of generation t (however, it does not guarantee that the whole path will be an equilibrium).

Suppose now that the generation t entrepreneurs are not financially constrained (i.e. we look for a solution of the system where $\lambda_t = 0$). Then, the set of conditions that determine the state of the economy at the next generation reduces to:

$$K_t(t+1) + Z_t(t+1) + B\left(t+1\right) = \left(1-\alpha-\beta\right)Y_{t-1}\left(t+1\right)$$

$$Z_t(t+1) = \frac{\beta}{\alpha}K_t(t+1)$$

$$B\left(t+1\right)=R\left(t+1\right)B\left(t\right)$$

$$Y_t(t+2)=A(t+2)K_t(t+1)^{\alpha}Z_t(t+1)^{\beta}$$

$$R(t+2)=\alpha\frac{Y_t(t+2)}{K_t(t+1)}$$

The state of the system when the financing constraint is not binding is determined as follows. Given $[Y_{t-1}(t+1), B(t+1)]$, $Z_t(t+1)$ and $K_t(t+1)$ are jointly determined by the first two equations.

This determines $Y_t(t+2)$ and $R(t+2)$ through the last two equations, and in turn this determines the state vector in the next generation, $[Y_t(t+2), B(t+2)]$.

A necessary condition for the existence of the solution to these equations is that:

$$B\left(t+1\right)\leq\left(1-\alpha-\beta\right)Y_{t-1}\left(t+1\right)=B_{max,nb}(Y_{t-1}(t+1))$$

which again imposes that the bubble cannot be too large.

Another necessary condition is that the constraint is slack:

$$Z_t(t+1)\leq\pi(1-\alpha-\beta)Y_{t-1}(t+1)$$

i.e.

$$\left(1-\alpha-\beta\right)Y_{t-1}\left(t+1\right)-B\left(t+1\right)\leq\pi(1-\alpha-\beta)Y_{t-1}(t+1)(1+\frac{\alpha}{\beta})$$

$$B\left(t+1\right)\geq\left[(1-\pi)-\frac{\alpha}{\beta}\pi\right]\left(1-\alpha-\beta\right)Y_{t-1}\left(t+1\right)=B_{max,b}(Y_{t-1}(t+1))=B_{min,nb}(Y_{t-1}(t+1))$$

Therefore, there exists a nonbinding continuation if and only if:

$$B_{max,b}(Y_{t-1}(t+1)) \leq B(t+1) \leq B_{max,nb}(Y_{t-1}(t+1))$$

This condition is also sufficient for the existence to the intratemporal nonbinding equilibrium of generation t (however, it does not guarantee that the whole path will be an equilibrium). It is also clear that this condition can be satisfied only in the NB region, so that a nonbinding continuation cannot occur in the B region.

From the analysis above, it becomes clear that given a value of the state vector $[Y_{t-1}(t+1), B(t+1)]$, whether the continuation is binding or not is uniquely determined by whether we are in the B or NB regions of the parameter space. Once the state of the constraint in the continuation is known,$Y_{t-1}(t+1)$ and $B(t+1)$ are sufficient to compute all the equilibrium prices and quantities in the next period and, consequently, represent a state vector for the economy.

It is immediate to see that $[\hat{Y}_{t-1}(t+1), B^{*}(t+1)]$ is also a state vector for the dynamics of the economy and the threshold value of the bubble for the binding and nonbinding continuation can be expressed as:

$$B^{*}_{max,b} = \left[(1-\pi) - \frac{\alpha}{\beta}\pi\right](1-\alpha-\beta)$$

Proof of Lemma 3

The AMSZ criterion requires that for each t

$$Y_t(t+2) - (1-\alpha-\beta)\, Y_t(t+2) \geq (1-\alpha-\beta)\, Y_t(t+2)$$

since $Y_t(t+2)$ is the amount of resources produced, $1-\alpha-\beta$ is the fraction of it paid to labor (the rest is payed as returns to capital), and total wages $(1-\alpha-\beta)Y_t(t+2)$ are equivalent to aggregate savings by generation $t+1$. Simplifying, we get $(\alpha+\beta) \geq \frac{1}{2}$.

Based on the equations for the steady-state presented in the proof of proposition 1, we have

$$\pi IRR + (1-\pi) R = \frac{\alpha+\beta}{(1-\alpha-\beta)} (1+g) \geq (1+g)$$

independently of whether the economy is in the B or in the NB region.

Proof of Proposition 3

Consider an economy with state $\left[\hat{Y}_{t-1}(t+1), B^*(t+1)\right]$ such that

$$0 < B^*(t+1) \leq (1-\pi)(1-\alpha-\beta) - \alpha$$

and

$$\hat{Y}_{t-1}(t+1) > 0$$

Since the economy belongs to the B region, it is easy to check that

$$B(t+1) \leq B_{max,b}(Y_{t-1}(t+1))$$

Based on Lemma 2, we know the borrowing constraints will bind at $t+1$. Hence, the equilibrium in this period is characterized by:

$$K_t(t+1) + B(t+1) = (1-\pi)(1-\alpha-\beta) Y_{t-1}(t+1)$$

$$Z_t(t+1) = \pi (1-\alpha-\beta) Y_{t-1}(t+1)$$

$$Y_t(t+2) = A(t+2) K_t(t+1)^{\alpha} Z_t(t+1)^{\beta}$$

$$R(t+2) = \alpha \frac{Y_t(t+2)}{K_t(t+1)}$$

and the bubble will evolve according to

$$B(t+2) = R(t+2)\, B(t+1)$$

Substituting the savings conditions on the production function and rearranging terms, we obtain:

$$\hat{Y}_t(t+2) = (1+g)^{-(\alpha+\beta)} \left[\pi(1-\alpha-\beta)\right]^{\alpha} \left[(1-\pi)(1-\alpha-\beta) - B^*(t+1)\right]^{\beta} \hat{Y}_{t-1}(t+1)^{\alpha+\beta}$$

Combining the savings condition with the expressions for the interest rate and the bubble dynamics, we obtain:

$$B^*(t+2) = \alpha \left[\frac{B^*(t+1)}{(1-\pi)(1-\alpha-\beta) - B^*(t+1)}\right]$$

These two equations fully describe the dynamics of the economy.

If

$$B^*(t+1) = (1-\pi)(1-\alpha-\beta) - \alpha$$

the solution to the system of difference equations is given by:

$$B^*(t+j) = (1-\pi)(1-\alpha-\beta) - \alpha$$

$$\hat{Y}_{t+j-1}(t+j+1) = \left\{ \frac{\left[\pi(1-\alpha-\beta)\right]^{\alpha} \alpha^{\beta}}{(1+g)^{\alpha+\beta}} \right\} \hat{Y}_{t+j-2}(t+j)^{\alpha+\beta}$$

for all $j = \{1, 2, 3...\}$. It is easy to verify that this economy converges to a unique steady-state characterized by:

$$\hat{Y} = \left\{ \frac{\left[\pi(1-\alpha-\beta)\right]^{\alpha} \alpha^{\beta}}{(1+g)^{\alpha+\beta}} \right\}^{\frac{1}{1-\alpha-\beta}}$$

$$B^* = (1-\pi)(1-\alpha-\beta) - \alpha$$

$$\hat{Z} = \frac{\pi(1-\alpha-\beta)}{(1+g)}\hat{Y}$$

$$\hat{K} = \frac{\alpha}{(1+g)}\hat{Y}$$

If, on the other hand,

$$B^*(t+1) < (1-\pi)(1-\alpha-\beta) - \alpha$$

the bubble to output ratio converges to zero and the economy converges to the bubbleless binding steady-state characterized in the proof of Proposition 2.

Proof of Proposition 4

That no bubble can exist before T is a trivial result, since $ASMZ$ is satisfied and the economy is never expected to move to the B region. Now, suppose that, at the very beginning of date T, agents learn about a permanent change in (α, β) to (α', β'). Naturally, if bubbles are to be sustained from that point on and if the $AMSZ$ is satisfied, financial constraints have to bind and we must have:

$$\frac{\beta'}{\alpha'} > \left[\frac{\pi}{1-\pi}\right]$$

and

$$\alpha' + \beta' > 0.5$$

Suppose, additionally, that the following holds

$$[(1-\alpha'-\beta')(1-\pi) - \alpha'] > 0$$

It is straightforward to check that, for any values of $(\alpha' + \beta')$ in the interval$(0.5, 1)$ and any $\pi \in (0, 1)$, those conditions will be satisfied provided that $\frac{\beta'}{\alpha'}$ is large enough.

We will focus on an equilibrium in which the bubble is issued uniformly at time T by entrepreneurs of generation $T-1$, who will immediately sell it to the households of the same generation. Naturally, from that point on, the rational bubble will evolve at the market interest rate and will be traded among households of different generations only. We also assume that the bubble issued will be the largest possible.

Because in this equilibrium the bubble relaxes the borrowing constraint of the entrepreneurs of generation $T-1$, it allows them to invest more in intangible capital. If the size of the bubble is large enough, the entrepreneur might have enough resources to invest as much as he wants in intangible capital, i.e. the financing constraint becomes slack.

Instead, the effect of this bubble on the future generations comes entirely from the increased production of those entrepreneurs. From time $T+1$ on, the dynamics of the economy follow exactly those derived in Lemma 2 and Proposition 3. In particular, an equilibrium in which a bubble is created at time T requires that the bubble never grows beyond the maximum valued reported in the proof of Proposition 3.

Focus then on an equilibrium in which at $T+1$ the bubble is positive but small enough that the dynamic equilibrium can be sustained (The analysis of Proposition 3 shows that under the conditions assumed above, such a bubble can always be found). Call this level of the bubble $\overline{B}(t+1)$.

Now, consider the required size of the bubble at time T, $B(T)$, such that it will grow exactly to $\overline{B}(t+1)$ between t and $t+1$. It must satisfy:

$$\overline{B}\left(T+1\right) = R\left(T+1\right) B\left(T\right)$$

In order to determine $R(T+1)$, we need to solve for the intratemporal equilibrium at time T. In particular, it can be that the bubble $B(T)$ will be small enough that the financing constraint is still binding for the entrepreneur, i.e. the system satisfies the equations

$$K_{T-1}(T) + \frac{\overline{B}(T+1)}{R(T+1)} = (1-\pi)(1-\alpha-\beta)\,Y_{T-2}(T)$$

$$Z_{T-1}(T) = \pi(1-\alpha-\beta)\,Y_{T-2}(T)$$

$$Y_{T-1}(T+1) = A(T+1)\,K_{T-1}(T)^{\alpha}\,Z_{T-1}(T)^{\beta}$$

$$R(T+1) = \alpha\frac{Y_{T-1}(T+1)}{K_T(T+1)}$$

or it is slack, i.e. it satisfies:

$$K_{T-1}(T) + \frac{\overline{B}(T+1)}{R(T+1)} = (1-\pi)(1-\alpha-\beta)\,Y_{T-2}(T)$$

$$Z_{T-1}(T) = \frac{\alpha}{\beta}K_{T-1}(T)$$

$$Y_{T-1}(T+1) = A(T+1)\,K_{T-1}(T)^{\alpha}\,Z_{T-1}(T)^{\beta}$$

$$R(T+1) = \alpha\frac{Y_{T-1}(T+1)}{K_T(T+1)}$$

As shown in the derivation of Lemma 2, either one or the other system will have a solution, given the required $\overline{B}(T+1)$ and the initial conditions $Y_{T-2}(T)$, provided that the implied bubble $B(T)$ is not too large. In turn, the relevant system of equations determines the output and the bubble at time T. Therefore, for a small enough bubble $\overline{B}(T+1)$ we can sustain an equilibrium in which the bubble is issued by the entrepreneurs at time T, does not exist before, and persists from time T on.

Proof of Proposition 5

Suppose that, upon a technological change, the new parameters (α', β') are such that:

$$[(1-\alpha'-\beta')(1-\pi)-\alpha'] > 0$$

$$\frac{\beta'}{\alpha'} > \frac{\pi}{1-\pi}$$

$$\beta' + \pi(\alpha'+\beta') > \pi$$

It is straightforward to check that, for any values of $(\alpha'+\beta')$ in the interval$(0.5, 1)$ and any $\pi \in (0,1)$, those conditions will be satisfied provided that $\frac{\beta'}{\alpha'}$ is large enough.

We will prove the existence of one type of bubbly equilibria, where the bubble is a constant fraction of output until technological change occurs, and jumps when it happens, growing at the market interest rate thereafter. To start, suppose the economy arrives at date $t+1$ with initial condition $[Y_{t-1}(t+1), B(t+1)]$ and no technological progress has occurred yet. We assume that $B(t+1) = \theta Y_{t-1}(t+1)$, where $0 < \theta < 1$. Suppose that, if technological progress occurs in period $t+2$, the bubble jumps to

$$\tilde{B}(t+2) = \psi Y_t(t+2)$$

where

$$\psi = [(1-\alpha'-\beta')(1-\pi)-\alpha']\left(\frac{1-\alpha-\beta}{1-\alpha'-\beta'}\right)$$

The restrictions on the parameters guarantee that $0 < \psi < 1$.

This choice of $\tilde{B}(t+2)$ ensures that, if technological change takes place at the beginning of period $t+2$, the new economy will be in a bubbly equilibrium that converges to a bubbly steady-state similar to the one defined in proposition 3. To confirm this, first we check that, for this choice of the bubble and conditional on a technological shift, the borrowing constraint binds at date $t+2$. Suppose, for the sake of contradiction, that this is not the case. Then, we have

$$K_{t+1}(t+2) + Z_{t+1}(t+2) + \tilde{B}(t+2) = (1-\alpha-\beta) Y_t (t+2)$$

$$Z_{t+1}(t+2) < \pi (1-\alpha-\beta) Y_t (t+2)$$

$$K_{t+1}(t+2) = \frac{\alpha'}{\beta'} Z_{t+1}(t+2)$$

Note that the labor income received by the middle age entrepreneurs at date T is still determined by the parameters of the old technology. Combining the three conditions and the expression for $\tilde{B}(t+2)$, we get

$$\beta' + \pi(\alpha' + \beta') < \pi$$

which violates the initial assumption about the parameters. Hence, the borrowing constraint binds at time $t+2$.

Second, we follow the steps in the proof of proposition 3 and verify that, upon technological progress, the dynamic equations characterizing the transition of the economy from $t+2$ to $t+3$ reduce to:

$$Y_{t+1}(t+3) = A(t+3) \left[\alpha' \left(\frac{1-\alpha-\beta}{1-\alpha'-\beta'} \right) \right]^{\alpha'} \left[\pi (1-\alpha-\beta) \right]^{\beta'} Y_t (t+2)^{\alpha'+\beta'}$$

$$\tilde{B}^*(t+3) = \alpha' \frac{\tilde{B}^*(t+2)}{\left[(1-\alpha-\beta)(1-\pi) - \tilde{B}^*(t+2) \right]}$$

which yields:

$$\tilde{B}^*(t+3) = \left[(1-\alpha'-\beta')(1-\pi) - \alpha' \right]$$

Because, from time $t+2$ on, the technology is constant, it is clear that this economy will evolve as the economy in the proof of proposition 3. Hence, it will converge to a bubbly steady-state and financing constraints will bind at every period. Therefore, we have shown that, upon technological change occurring at date $t+2$, the proposed allocation is an equilibrium thereafter.

If technological change does not occur at the beginning of period $t+2$, the economy will arrive at that period with initial condition $[Y_t(t+2), B(t+2)]$ where $B(t+2) = \theta Y_t(t+2)$ again. Therefore, if the proposed allocation is an equilibrium from a time $t+1$ perspective for any $Y_{t-1}(t+1) > 0$, it will be an equilibrium from time a $t+1+j$ perspective as well, for any $j = \{1, 2, 3...\}$. Hence, to finalize the proof we need to show the proposed allocation is indeed an equilibrium at time $t+1$.

Since technological progress has not occurred at time $t+1$, entrepreneurs know they will produce using the old technology and everybody anticipates the market interest rate. Additionally, the economy is in the NB region of the parameter space, which implies that, for any positive bubble, financing constraints do not bind. Hence, the equilibrium in that period requires that:

$$K_t(t+1) + Z_t(t+1) + B(t+1) = (1-\alpha-\beta) Y_{t-1}(t+1)$$

$$K_t(t+1) = \frac{\alpha}{\beta} Z_t(t+1)$$

Manipulation of these equilibrium conditions yields:

$$K_t(t+1)\left(\frac{\alpha+\beta}{\alpha}\right) = (1-\alpha-\beta) Y_{t-1}(t+1) - B(t+1)$$

which is equivalent to

$$\frac{Y_{t-1}(t+1)}{K_t(t+1)} = \frac{\alpha+\beta}{\alpha\left[(1-\alpha-\beta)-\theta\right]}$$

Moreover, the evolution of the bubble requires that

$$q\tilde{B}\left(t+2\right)+\left(1-q\right)B\left(t+2\right)=R\left(t+2\right)B\left(t+1\right)$$

which reduces to

$$q\psi+\left(1-q\right)\theta=\alpha\theta\frac{Y_{t-1}\left(t+1\right)}{K_{t}\left(t+1\right)}$$

or

$$q\psi+\left(1-q\right)\theta=\frac{\theta\left(\alpha+\beta\right)}{\left[\left(1-\alpha-\beta\right)-\theta\right]}$$

The right hand side of this expression varies from zero to infinity as θ varies from zero to $(1-\alpha-\beta)$, whereas the left hand side varies from $q\psi$ to $q\psi+(1-q)(1-\alpha-\beta)$ which is always between $(0,1)$. Hence, there exists a $\theta^{*}\in(0,1-\alpha-\beta)$ which solves the equation above.

Therefore, for any $Y_{t-1}(t+1)>0$, the proposed process for the bubble is indeed an intertemporal stochastic equilibrium. As a matter of fact, there are infinitely many more equilibria. For instance, if we choose $\psi^{*}<\psi$ and adjust θ^{*} appropriately, we will have equilibria where, upon technological change, the economy is in a bubbly path that converges to an steady-state where the bubble-to-output ratio is zero, as in proposition 3.

Proof of Proposition 6

Part i) is straightforward. Suppose that technological change has not taken place at date T. Hence, the economy is in the NB region of the parameter space forever and it also satisfies the $AMSZ$ condition. Based on proposition 2, there can be no bubbles from that date on.

As for part ii) , the proof follows very similar steps to the proof of proposition 6. So, suppose that the new parameters (α',β') are such that:

$$\left[\left(1-\alpha'-\beta'\right)\left(1-\pi\right)-\alpha'\right]>0$$

$$\frac{\beta'}{\alpha'} > \frac{\pi}{1-\pi}$$

$$\beta' + \pi\left(\alpha' + \beta'\right) > \pi$$

It is straightforward to check that, for any values of $(\alpha' + \beta')$ in the interval$(0.5, 1)$ and any $\pi \in (0,1)$, those conditions will be satisfied provided that $\frac{\beta'}{\alpha'}$ is large enough.

We derive a bubbly allocation that is an intratemporal equilibrium at time $T-1$, and show it is indeed part of an intertemporal equilibrium. We will focus on an equilibrium path in which, if technological change takes place, the bubble grows at the market interest rate thereafter. So suppose that, at date $T-1$, the economy has initial condition $[Y_{T-3}(T-1), B(T-1)]$. Since the economy originally belongs to the NB region, and given that the middle age entrepreneurs of generation $T-2$ know for sure the technological specification they use to produce between $T-1$ and T, it is clear that financing constraints are slack. Hence, the intratemporal equilibrium at time $T-1$ is characterized by:

$$K_{T-2}(T-1) + Z_{T-2}(T-1) + B(T-1) = (1-\alpha-\beta)\, Y_{T-3}(T-1)$$

$$Z_{T-2}(T-1) = \frac{\beta}{\alpha} K_{T-2}(T-1)$$

$$Y_{T-2}(T) = A(T)\, K_{T-2}(T-1)^{\alpha}\, Z_{T-2}(T-1)^{\beta}$$

$$R(T) = \alpha \frac{Y_{T-2}(T)}{K_{T-2}(T-1)}$$

Moreover, based on i), the bubble has to evolve according to

$$\tilde{B}(T) = \frac{R(T)\, B(T-1)}{q}$$

if technology shifts and

$$B(T) = 0$$

otherwise.

Let us define $\tilde{B}(T)$ as in the proof of proposition 6:

$$\tilde{B}(T) = \psi Y_{T-2}(T)$$

where

$$\psi = \left[(1-\alpha'-\beta')(1-\pi) - \alpha'\right]\left(\frac{1-\alpha-\beta}{1-\alpha'-\beta'}\right)$$

The restrictions on the parameters guarantee that $0 < \psi < 1$.

As shown in the proof of proposition 6, this choice of $\tilde{B}(T)$ ensures that, if technological change takes place, the new economy will be in a bubbly equilibrium that converges to a bubbly steady-state similar to the one defined in proposition 3. To prove that, from a time $T-1$ perspective, the allocation is indeed part of an unconditional intertemporal equilibrium - taking into account uncertainty about technological change at T - we need to show that the set of admissible initial conditions $[Y_{T-3}(T-1), B(T-1)]$ is not empty. So, without loss of generality, let us assume that $B(T-1) = \theta Y_{T-3}(T-1)$. We now show that $\theta \in (0, 1-\alpha-\beta)$. This is important, otherwise the proposed bubble would outgrow savings in period $T-1$, which is not feasible.

Note that the no-arbitrage relationship for the bubble requires that

$$\tilde{B}(T) = \frac{R(T)\,B(T-1)}{q}$$

which is equivalent to

$$q\psi = \alpha \frac{B(T-1)}{K_{T-2}(T-1)}$$

Since the economy is in the NB region at date $T-1$, we know that financing constraints are slack, which results in

$$K_{T-2}\left(T-1\right)=\left(\frac{\alpha}{\alpha+\beta}\right)\left[\left(1-\alpha-\beta\right)Y_{T-3}\left(T-1\right)-B\left(T-1\right)\right]$$

This condition generates:

$$\frac{Y_{T-3}\left(T-1\right)}{K_{T-2}\left(T-1\right)}=\frac{\alpha+\beta}{\alpha\left[\left(1-\alpha-\beta\right)-\theta\right]}$$

Hence, we have:

$$q\psi=\frac{\left(\alpha+\beta\right)\theta}{\left[\left(1-\alpha-\beta\right)-\theta\right]}$$

or

$$\theta=\left[\frac{q\psi}{q\psi+\alpha+\beta}\right]\left(1-\alpha-\beta\right)$$

Since $\theta\in\left(0,1-\alpha-\beta\right)$, we confirm that, for any $Y_{T-3}\left(T-1\right)>0$, the proposed process for the bubble is indeed an intertemporal stochastic equilibrium. As a matter of fact, there are infinitely many more equilibria. For instance, if we choose $\psi^{*}<\psi$ and adjust θ appropriately, we will have equilibria where, upon technological change, the economy is in a bubbly path that converges to an steady-state where the bubble-to-output ratio is zero, as in proposition 3.

Proof of Proposition 7

An economy in which the financing constraint is strictly binding and satisfies the AMSZ test will respect two conditions:

$$\left(\beta+\alpha\right)>\frac{1}{2}$$

$$\frac{\beta}{\alpha}>\frac{\pi}{\left(1-\pi\right)}$$

We show that in this region of the parameter space we can construct a deviation that does not rely on intragenerational transfers and improves the utility of the first generation without reducing the utility of any other individual. For simplicity, our counterexample focuses on an economy in the bubbleless steady state. We consider a deviation small enough that the economy remains in the binding region.

Call $\overline{K}_t(t+1)$ the amount of capital in the original allocation (which is the steady state of this economy), and similarly for all other variables. Start by transferring an amount dC of goods from the households of generation t to the old of generation $t-1$, at period $t+1$, just before production occurs. This generation needs then to be compensated by an amount of consumption (in period $t+2$) equal to the amount of consumption lost due to the transfer to generation $t-1$.

The loss in output for this generation will be:

$$d\tilde{Y}_{t+1} = f(\overline{K}_t(t+1), \overline{Z}_t(t+1)) - f(\overline{K}_t(t+1) - dC, \overline{Z}_t(t+1))$$

The first-order linear approximation of this loss will be:

$$dY_{t+1} = \overline{R}dC$$

where $\overline{R} = \overline{R}(t+1) = \frac{\partial f(\overline{K}_t(t+1), \overline{Z}_t(t+1))}{\partial K}$ which is constant in the steady state. Importantly, note that since f is concave with respect to K, we have:

$$d\tilde{Y}_{t+1} \leq dY_{t+1}$$

locally, which means that if we compensate by giving dY_{t+1} instead of $d\tilde{Y}_{t+1}$ to generation $t+1$, they will actually be made better off (by a second-order term).

Assume then that we transfer $(\alpha+\beta)dY_{t+1}$ from generation $t+2$ households to generation $t+1$ households. Note that the fact that generation $t+1$ invests less capital means that output in $t+2$ will be lower, which in turn means that the young of generation $t+2$, who provide labor and get a share $(1-\alpha-\beta)$ of output at time $t+3$, will be losing an amount $(1-\alpha-\beta)dY_{t+1}$ of resources. The lower starting wealth of the young will affect both those who become households and those who become entrepreneurs, in proportion $(1-\pi)$ and π respectively. Since we are still in the binding region, this will imply that production of this generation will occur with an amount of aggregate physical capital lowered by $(1-\alpha-\beta)(1-\pi)dY_{t+1}$ and an amount of intangible capital lowered by $(1-\alpha-\beta)\pi dY_{t+1}$. Therefore, the loss of output to generation $t+2$ will be:

$$d\tilde{Y}_{t+2} = f(\overline{K}_{t+1}(t+2), \overline{Z}_{t+1}(t+2))$$

$$-f(\overline{K}_{t+1}(t+2) - (1-\alpha-\beta)(1-\pi)dY_{t+1} - (\alpha+\beta)dY_{t+1}, \overline{Z}_{t+1}(t+2) - (1-\alpha-\beta)\pi dY_{t+1})$$

The first order approximation to the total loss of output for generation $t+2$ is again enough for our purposes due to the concavity of the production function. Therefore, we can (more than) compensate generation $t+2$ by transferring to them an amount:

$$dY_{t+2} = \{R[(1-\alpha-\beta)(1-\pi) + (\alpha+\beta)] + IRR(1-\alpha-\beta)\pi\}\, dY_{t+1}$$

or

$$dY_{t+2} = \{[\pi IRR + (1-\pi)R](1-\alpha-\beta) + (\alpha+\beta)R\} dY_{t+1}$$

The same reasoning holds for the future generations. Therefore, the following difference equation holds generally for $s > 1$:

$$dY_{t+s} = \{[\pi IRR + (1-\pi)R](1-\alpha-\beta) + (\alpha+\beta)R\} dY_{t+s-1}$$

Re-expressing the equation in relative changes, we obtain:

$$\frac{dY_{t+s}}{Y_{t+s}} = \frac{\{[\pi IRR + (1-\pi)R](1-\alpha-\beta) + (\alpha+\beta)R\}}{1+g} \frac{dY_{t+s-1}}{Y_{t+s-1}}$$

The deviation will be feasible as long as this difference equation converges to a number less than one, where $\frac{dY}{Y}$ is constant. Convergence will occur if and only if:

$$[\pi IRR + (1-\pi)R](1-\alpha-\beta) + (\alpha+\beta)R < 1+g$$

To finish the proof, we just need to find a region of the parameter space $\{\alpha, \beta, \pi\}$ for which this equation is satisfied together with

$$(\beta+\alpha) > \frac{1}{2}$$

$$\frac{\beta}{\alpha} > (1-\pi)$$

that guarantee that the AMSZ test is satisfied and that we are in the binding region. By rewriting the convergence equation as:

$$(\alpha+\beta)(1+g) + \frac{(1+g)\alpha}{(1-\pi)(1-\alpha-\beta)} \leq 1+g$$

it is easy to see how it is always possible to achieve that result by choosing α low enough and β high enough.

Proof of Proposition 8

The proof mimics the proof of Proposition 1 in AMSZ, taking into account that because $IRR > R$, in any intertemporal transfer scheme entrepreneurs should be compensated more than households, which forces the planner to transfer IRR to both groups (since she cannot distinguish between them).

Examples and extensions

Stochastic bubbles

This section presents a model with stochastic bubbles. For continuity of exposition, proofs are reported at the end of the section. Consider an environment in which, at every period, the rational bubble survives with probability p or bursts with probability $1-p$ as in the classic model of Blanchard and Watson (1982). Naturally, as long as the bubble exists, it has to grow at a rate $\frac{R}{p}$, so its expected return is equal to R.

The definition of a conditional steady-state is the first step in developing the machinery to analyze stochastic bubbly equilibria.

Definition. *A conditional steady-state is an allocation such that a stochastic bubble exists and then bursts at some point. While the bubble exists, all variables in the economy* $\left\{\hat{Y}_t(t+2), \hat{K}_t(t+1), \hat{Z}_t(t+1), B^*(t+1)\right\}$ *are constant. After the bubble bursts, the economy converges to a bubbleless steady-state as shown in subsection 4.1.*

Naturally, unless the economy starts at the conditional steady-state, it will almost surely not reach it, since the probability of that event happening converges to zero. The conditional steady-state is, notwithstanding, a useful benchmark because it determines the trajectory of a bubbly economy while the bubble lasts. In what follows, a series of propositions characterizes the dynamics of the economy when bubbles are stochastic. They focus on the case where the stochastic bubble has not burst until the most recent period, since the dynamics of the economy after the bubble bursts has been explored in Section 4.1 of the text already.

Lemma 1. *Along any bubbly path with initial condition* $[Y_{t-1}(t+1), B(t+1)]$*, all the results in Lemma 2 of the text hold.*

Lemma 1 shows that the dynamics of the economy are not materially affected by the stochastic nature of bubbles. Whether the continuation is binding or not depends only on the bubble-to-output ratio and the region of the parameter space where the economy lies.

Proposition 1. *There are no stochastic bubbly equilibria in the NB region if AMSZ is satisfied.*

As before, economies that are productive in the AMSZ sense cannot sustain stochastic bubbly equilibria if they are in the NB region of the parameter space.

Proposition 2. *In the B region with $\alpha + \beta \geqslant \frac{1}{2}$, there exist dynamic bubbly equilibria that converge to a conditional steady-state in which*

$$B^* = (1 - \alpha - \beta)(1 - \pi) - \frac{\alpha}{p} \tag{1}$$

and the borrowing constraint is binding at all times, as long as $B^* > 0$. There also exist dynamic equilibria such that the bubble-to-output ratio is always non-negative but converges to zero over time. \

These results show that economies that satisfy the AMSZ criterion can sustain stochastic bubbles. Obviously, the conditions for the existence of stochastic bubbles are more stringent relative to the deterministic case. As mentioned before, conditional on not bursting, stochastic bubbles grow at the rate $\frac{R(t+1)}{p} > R(t+1)$. Hence, for these bubbles to not outgrow the economy in finite time, interest rates in the bubbleless equilibrium have to be even lower compared to the economy's growth rate in the long-run (the more so the higher the chance the bubble will burst). This can be achieved by having a sufficiently high ratio $\frac{\beta}{\alpha}$ and binding financing constraints.

It is interesting to consider the macroeconomic consequences of a bursting bubble. The crash immediately reduces the consumption of the old households currently alive, whose portfolios were exposed to the bubble. Because the entrepreneurs never buy the bubbly security in equilibrium, they are unaffected. Hence, aggregate consumption is immediately reduced. Future output and investment, however, will expand. In the absence of a bubble, the middle-aged households of following generations have no option but to make collateralized loans to the contemporaneous entrepreneurs, raising the stock of physical capital and increasing production. This predicted positive impact is at odds with the historical experience, since asset market crashes tend to be followed by prolonged slumps in real activity. The model's inability to generate such effect results mainly from the lack of a direct link between the price of the bubbly asset and the balance sheet of levered agents, as instead is the case in Kiyotaki and Moore (1997).

This limitation can be solved by extending the baseline specification and considering an environment where the entrepreneurs' capacity to raise funds is directly affected by the bubble. More specifically, instead of taking the total amount of the bubble as given, assume that, at each period, the middle-aged entrepreneurs are able to issue an additional quantity of bubbly assets to be sold to the contemporaneous households. This process continues as long as the bubble hasn't burst, but stops permanently after a crash. Clearly, the continued issuance of bubbles alleviates financial frictions for all generations, allowing entrepreneurs to increase the amount of resources invested in intangible capital. The bursting of the bubble shuts down this channel, exacerbating financial frictions and increasing distortions in real investments for all future generations.[1] Depending on the strength of this effect, a crash in bubbly securities might reduce output and investment in the long-run. This possibility is investigated formally below, where the following Proposition is proved:

Proposition 3. *Call $Q(t)$ the quantity of the bubble at time t. Call $B(t)$ its price. Suppose every entrepreneur can issue an additional quantity $\eta Q(t-1)$ of bubble at t, and, if the bubble bursts, no entrepreneur can issue the bubble anymore. Then, there exists an area in the B region of the parameter space such that the bursting of the bubble will cause a permanent drop in output.*

[1] In fact, this mechanism is similar to the one explored in Kocherlakota (2009).

Stochastic bubbles - proofs

Proof of Appendix Lemma 1

Let us start with cases *ii*) and *iii*) of Lemma 2. The characterization of the slack continuation follows the proof of that Lemma. Conditional on the bubble not bursting until period $t+1$, the intratemporal equilibrium in the savings market at that date requires:

$$K_t(t+1) + Z_t(t+1) + B(t+1) = (1-\alpha-\beta)\,Y_{t-1}(t+1)$$

If financial constraints are to be slack, we must have:

$$Z_t(t+1) < \pi(1-\alpha-\beta)\,Y_{t-1}(t+1)$$

and

$$K_t(t+1) = \frac{\alpha}{\beta} Z_t(t+1)$$

A final restriction is that the bubble does not exhaust all savings of generation t

$$B(t+1) \leq (1-\alpha-\beta)\,Y_{t-1}(1+1)$$

Substituting these restrictions in the savings market equilibrium proves condition *iii*) in Lemma 2.

If, on the other hand, the constraints are to be binding, we must have:

$$Z_t(t+1) = \pi(1-\alpha-\beta)\,Y_{t-1}(t+1)$$

$$K_t(t+1) + B(t+1) = (1-\pi)(1-\alpha-\beta)\,Y_{t-1}(t+1)$$

Substituting these two conditions in the savings market equilibrium results in condition *ii*) in Lemma 2.

Case $i)$ can be proved by contradiction following the steps above. Assume $(\alpha, \beta, \pi, n) \subset NB$ but suppose that financing constraints bind at $t+1$. Hence

$$B(t+1) \leq \left[\frac{(1-\pi)}{\pi} - \frac{\alpha}{\beta}\right] \pi (1-\alpha-\beta) Y_{t-1}(t+1) < 0$$

which is an absurdity.

Finally, it follows that conditional on the bubble not bursting, $[Y_{t-1}(t+1), B(t+1)]$ is a state vector for the economy.

Proof of Appendix Proposition 1

Suppose $\alpha + \beta \geq \frac{1}{2}$ and assume that the economy is in a bubbly path. Because $\frac{\beta}{\alpha} < \frac{\pi}{1-\pi}$, we know that financing constraints are slack - if anything, the existence of the bubble relaxes financing constraints. The state of the economy at the beginning of period $t+1$ is given by $[Y_{t-1}(t+1), B(t+1)]$. Hence, the equilibrium in the continuation is characterized by:

$$K_t(t+1) + Z_t(t+1) + B(t+1) = (1-\alpha-\beta) Y_{t-1}(t+1)$$

$$K_t(t+1) = \frac{\alpha}{\beta} Z_t(t+1)$$

$$Y_t(t+2) = A(t+2) K_t(t+1)^{\alpha} Z_t(t+1)^{\beta}$$

$$R(t+2) = \alpha \frac{Y_t(t+2)}{K_t(t+1)}$$

and the bubble will evolve according to

$$B(t+2) = \frac{R(t+2) B(t+1)}{p}$$

Simple manipulation of these conditions yields

$$\hat{Y}_t\left(t+2\right)=\left(\frac{\alpha}{\beta}\right)^{\alpha}\hat{Z}_t\left(t+1\right)^{\alpha+\beta}$$

$$\hat{Z}_t\left(t+1\right)=\left(\frac{\beta}{\alpha+\beta}\right)\left[\frac{\left(1-\alpha-\beta\right)-B^*\left(t+1\right)}{\left(1+g\right)}\right]\hat{Y}_{t-1}\left(t+1\right)$$

Isolating $\hat{Z}_t\left(t+1\right)$ from the production function and substituting it in the last equation yields

$$\left[\frac{\left(1+g\right)}{\beta^{\frac{\beta}{\alpha+\beta}}\alpha^{\frac{\beta}{\alpha+\beta}}}\right]\left[\frac{\left(\alpha+\beta\right)}{\left(1-\alpha-\beta\right)-B^*\left(t+1\right)}\right]=\frac{\hat{Y}_{t-1}\left(t+1\right)}{\hat{Y}_t\left(t+2\right)^{\frac{1}{\alpha+\beta}}}$$

Manipulating the equation representing the dynamics of the bubble, we obtain:

$$B^*\left(t+2\right)=\left[\frac{\beta^{\frac{\beta}{\alpha+\beta}}\alpha^{\frac{\alpha}{\alpha+\beta}}}{p\left(1+g\right)}\right]B^*\left(t+1\right)\frac{\hat{Y}_{t-1}\left(t+1\right)}{\hat{Y}_t\left(t+2\right)^{\frac{1}{\alpha+\beta}}}$$

Using the condition above and taking logs yields:

$$log\left(B^*\left(t+2\right)\right)-logB^*\left(t+1\right)=-log\left(p\right)+log\left(\frac{\alpha+\beta}{\left(1-\alpha-\beta\right)-B^*\left(t+1\right)}\right)$$

Since the AMSZ criterion requires that $\alpha+\beta\geq\frac{1}{2}$, the last equation implies that, for any $p\in(0,1]$, the bubble-to-output ratio will reach $(1-\alpha-\beta)$ in finite time, fully absorbing the combined savings of households and entrepreneurs. Hence, with positive probability the economy will be in an explosive path, which cannot be an equilibrium.

Proof of Appendix Proposition 2

Consider an economy with initial condition $\left[\hat{Y}_{t-1}\left(t+1\right),B^*\left(t+1\right)\right]$ such that

$$B^*\left(t+1\right)\leq B^*_{max,b}\left(Y_{t-1}\left(t+1\right)\right)$$

and

$$\hat{Y}_{t-1}(t+1) > 0$$

Based on Lemma 4, we know the borrowing constraints will bind at $t+1$. Hence, the equilibrium in this period is characterized by:

$$K_t(t+1) + B(t+1) = (1-\pi)(1-\alpha-\beta)\,Y_{t-1}(t+1)$$

$$Z_t(t+1) = \pi(1-\alpha-\beta)\,Y_{t-1}(t+1)$$

$$Y_t(t+2) = A(t+2)\,K_t(t+1)^{\alpha}\,Z_t(t+1)^{\beta}$$

$$R(t+2) = \alpha\frac{Y_t(t+2)}{K_t(t+1)}$$

and the bubble will evolve according to

$$B(t+2) = \frac{R(t+2)\,B(t+1)}{p}$$

if it doesn't burst.

Substituting the savings conditions in the production function and rearranging terms, we obtain:

$$\hat{Y}_t(t+2) = (1+g)^{\alpha+\beta}\left[\pi(1-\alpha-\beta)\right]^{\alpha}\left[(1-\pi)(1-\alpha-\beta) - B^*(t+1)\right]^{\beta}\hat{Y}_{t-1}(t+1)^{\alpha+\beta}$$

Combining the savings condition with the expressions for the interest rate and the bubble dynamics, we obtain:

$$B^*(t+2) = \frac{\alpha}{p}\left[\frac{B^*(t+1)}{(1-\pi)(1-\alpha-\beta) - B^*(t+1)}\right]$$

These two equations fully describe the dynamics of the economy. Conditional on the bubble not bursting, there exists a solution to the system of difference equations given by:

$$B^*(t+j+1) = (1-\pi)(1-\alpha-\beta) - \frac{\alpha}{p}$$

$$\hat{Y}_{t+j-1}(t+j+1) = \left\{ \frac{[\pi(1-\alpha-\beta)]^{\alpha}\left[\frac{\alpha}{p}\right]^{\beta}}{(1+g)^{\alpha+\beta}} \right\} \hat{Y}_{t+j-2}(t+j)^{\alpha+\beta}$$

for all $j = \{1, 2, 3...\}$.

It is immediate to verify that, while the bubble lasts, this economy converges to a conditional steady-state characterized by:

$$\hat{Y} = \left\{ \frac{[\pi(1-\alpha-\beta)]^{\alpha}\left[\frac{\alpha}{p}\right]^{\beta}}{(1+g)^{\alpha+\beta}} \right\}^{\frac{1}{1-\alpha-\beta}}$$

$$\hat{Z} = \frac{\pi(1-\alpha-\beta)}{(1+g)}\hat{Y}$$

$$\hat{K} = \frac{\alpha}{p(1+g)}\hat{Y}$$

$$B^* = (1-\pi)(1-\alpha-\beta) - \frac{\alpha}{p}$$

$$R = p(1+g) = \alpha\frac{\hat{Y}}{\hat{K}}$$

Based on the restrictions imposed over the parameters, it is easy to check that this solution satisfies the condition

$$0 < B^* < B^*_{max,b}$$

Proof of Appendix Proposition 3

In our baseline case, the quantity of the bubble is fixed and its value is given by $B(t_0)$. Over time, the value of the bubble grows at a rate that makes households indifferent between holding it or investing in tangible capital, taking into account the risk of collapse of the bubble.

Just for illustration purposes, we can analyze the case in which the *quantity* of the bubble increases over time as well. In particular, we can imagine that the entrepreneurs of *each* generation are able to issue an additional amount of the bubble, at no cost, as long as the bubble has not burst. Note that, for the bubble to be attractive to households, its price must still grow at the usual rate, $\frac{R}{p}$.

Assume that the new quantity of the bubble $Q(t+1)$ that entrepreneurs from generation t can issue (at time $t+1$) is such that:

$$Q(t+1) = \eta Q(t)$$

The total value $V(t)$ of the bubble grows according to:

$$V(t+1) = B(t+1)Q(t+1) = V(t)\frac{R}{p}\eta$$

The main equations that describe the economy change as follows. The savings market equilibrium becomes:

$$K_t\,(t+1) + Z_t\,(t+1) + Q(t)B\,(t+1) = (1-\alpha-\beta)\,Y_{t-1}\,(t+1)$$

since savings must be invested in tangible capital, or intangible capital, or to buy the existing bubble (the issuance of the new bubble is a transfer within the same generation).

If financial constraints are to be slack, we must have:

$$Z_t\left(t+1\right) < \pi\left(1-\alpha-\beta\right)Y_{t-1}\left(t+1\right)+(Q(t+1)-Q(t))B(t+1) = \pi\left(1-\alpha-\beta\right)Y_{t-1}\left(t+1\right)+(\eta-1)Q(t)B(t$$

and

$$K_t\left(t+1\right) = \frac{\alpha}{\beta}Z_t\left(t+1\right)$$

If, on the other hand, the constraints are to be binding, we must have:

$$Z_t\left(t+1\right) = \pi\left(1-\alpha-\beta\right)Y_{t-1}\left(t+1\right) + (\eta-1)Q(t)B(t+1)$$

and

$$K_t\left(t+1\right) + (\eta-1)Q(t)B(t+1) + Q(t)B\left(t+1\right) = \left(1-\pi\right)\left(1-\alpha-\beta\right)Y_{t-1}\left(t+1\right)$$

or

$$K_t\left(t+1\right) + V\left(t+1\right) = \left(1-\pi\right)\left(1-\alpha-\beta\right)Y_{t-1}\left(t+1\right)$$

A final restriction is that the bubble does not exhaust all savings of generation t

$$V\left(t+1\right) \leq \left(1-\alpha-\beta\right)Y_{t-1}\left(1+1\right)$$

To see the effect of the bubble bursting, one can focus on the steady state of the economy (for simplicity, we only look at the steady state in which financial constraints are binding). Since $(\eta-1)Q(t)B(t+1) = (\eta-1)\frac{Q(t+1)}{\eta}B(t+1) = \frac{(\eta-1)}{\eta}V(t+1)$, we can rewrite all the equations in terms of V:

$$Y_t\left(t+2\right) = A\left(t+2\right)K_t\left(t+1\right)^{\alpha}Z_t\left(t+1\right)^{\beta}$$

$$R\left(t+2\right)=\alpha\frac{Y_t\left(t+2\right)}{K_t\left(t+1\right)}$$

$$Z_t\left(t+1\right)=\pi\left(1-\alpha-\beta\right)Y_{t-1}\left(t+1\right)+\frac{\left(\eta-1\right)}{\eta}V(t+1)$$

$$K_t\left(t+1\right)+V\left(t+1\right)=\left(1-\pi\right)\left(1-\alpha-\beta\right)Y_{t-1}\left(t+1\right)$$

$$V\left(t+2\right)=\eta\frac{R\left(t+2\right)}{p}V\left(t+1\right)$$

Rescaled as in Proposition 1, these equations yield

$$\hat{Y}_t\left(t+2\right)=\hat{K}_t\left(t+1\right)^{\alpha}\hat{Z}_t\left(t+1\right)^{\beta}$$

$$R(t+2)=\alpha\frac{\hat{Y}_t(t+2)}{\hat{K}_t(t+1)}$$

$$\hat{Z}_t\left(t+1\right)=\pi\frac{\left(1-\alpha-\beta\right)}{1+g}\hat{Y}_{t-1}\left(t+1\right)+\frac{\left(\eta-1\right)}{\eta(1+g)}V^{*}(t+1)\hat{Y}_{t-1}(t+1)$$

$$\hat{K}_t\left(t+1\right)+\frac{V^{*}\left(t+1\right)\hat{Y}_{t-1}(t+1)}{1+g}=\left(1-\pi\right)\frac{\left(1-\alpha-\beta\right)}{1+g}\hat{Y}_{t-1}\left(t+1\right)$$

$$V^{*}\left(t+2\right)=\eta\frac{R\left(t+2\right)}{p}\frac{V\left(t+1\right)}{Y_t(t+2)}=\eta\frac{R\left(t+2\right)}{p}V^{*}(t+1)\frac{Y_{t-1}\left(t+1\right)}{Y_t(t+2)}$$

In the steady state, $\hat{Y}, \hat{K}$ and $\hat{Z}$ are constant, so the respective level variables grow at rate $1+g$. If the bubble is to survive in steady state, it will be constant as a fraction of output, so that V^{*} will be constant as well. This steady state will satisfy:

$$\hat{Y}=\hat{K}^{\alpha}\hat{Z}^{\beta}$$

$$R=\alpha\frac{\hat{Y}}{\hat{K}}$$

$$\hat{Z}=\left[\pi\frac{\left(1-\alpha-\beta\right)}{1+g}+\frac{\left(\eta-1\right)}{\eta(1+g)}V^{*}\right]\hat{Y}$$

$$\hat{K}+\frac{V^{*}\hat{Y}}{1+g}=\left(1-\pi\right)\frac{\left(1-\alpha-\beta\right)}{1+g}\hat{Y}$$

$$1+g=\eta\frac{R}{p}$$

The last equation determines R:

$$R=(1+g)\frac{p}{\eta}$$

Then we have:

$$\frac{\hat{K}}{\hat{Y}}+\frac{V^*}{1+g}=(1-\pi)\frac{(1-\alpha-\beta)}{1+g}$$

$$\frac{V^*}{1+g}=(1-\pi)\frac{(1-\alpha-\beta)}{1+g}-\frac{\alpha}{R}$$

$$V^*=(1-\pi)(1-\alpha-\beta)-\frac{\alpha\eta}{p}$$

Plus

$$\hat{Y}=\hat{K}^{\alpha}\hat{Z}^{\beta}$$

$$R=\alpha\frac{\hat{Y}}{\hat{K}}$$

$$\hat{Z}=\left[\pi\frac{(1-\alpha-\beta)}{1+g}+\frac{(\eta-1)}{\eta(1+g)}V^*\right]\frac{1}{\alpha}R\hat{K}$$

So:

$$\hat{Y}=\hat{K}^{\alpha+\beta}\left\{\left[\pi\frac{(1-\alpha-\beta)}{1+g}+\frac{(\eta-1)}{\eta(1+g)}V^*\right]\frac{R}{\alpha}\right\}^{\beta}$$

Manipulating these expressions, we find that the bubbly steady state solves the following three equations (superscript B denotes the bubbly equilibrium):

$$\hat{Y}^B=\left[\left[\frac{\alpha\eta}{(1+g)p}\right]^{\alpha}\left\{\left[\pi\frac{(1-\alpha-\beta)}{1+g}+\frac{(\eta-1)}{\eta(1+g)}V^*\right]\right\}^{\beta}\right]^{\frac{1}{1-\alpha-\beta}}$$

$$R^B=(1+g)\frac{p}{\eta}$$

$$V^*=(1-\pi)(1-\alpha-\beta)-\frac{\alpha\eta}{p}>0$$

We can now compare this steady state with the bubbleless counterpart (to which the economy converges after the bubble bursts: superscript L). The bubbleless steady state solves:

$$\hat{Y}^L = \left[\left(\frac{\pi}{(1-\pi)} \right)^{\beta} ((1-\pi)(1-\alpha-\beta))^{\alpha+\beta} (1+g)^{-(\alpha+\beta)} \right]^{\frac{1}{1-\alpha-\beta}}$$

$$R^L = \frac{\alpha(1+g)}{(1-\pi)(1-\alpha-\beta)}$$

It is clear that the interest rate is surely higher in the equilibrium with the bubble. This results from two effects. First, the accumulated bubble crowds out investment in tangible capital, increasing R^B. Second, the issuance of additional bubble raises the investment in intangible capital, increasing the marginal productivity of tangible capital.

More formally, start from the condition that the bubble is nonnegative in steady state as a fraction of output:

$$V^* = (1-\pi)(1-\alpha-\beta) - \frac{\alpha\eta}{p} > 0$$

and rearrange:

$$\frac{\alpha}{(1-\pi)(1-\alpha-\beta)} < \frac{p}{\eta}$$

so it is clear that

$$R^L < R^B$$

Output in the bubbleless steady state can be higher or lower than in the bubbly steady state. The following Proposition proves that, depending on parameter values, output will *fall* in the steady state after the bubble bursts.

Proposition. *For α small enough relative to β, there exist parameter values in the B region such that $\hat{Y}^L < \hat{Y}^B$. Therefore, the bursting of the bubble can cause a permanent drop in output.*

Proof. Note that we can rewrite the inequality as:

$$\left(\left(1-\pi\right)\left(1-\alpha-\beta\right)\right)^{\alpha}\left(\pi\left(1-\alpha-\beta\right)\right)^{\beta}<\left[\frac{\alpha\eta}{p}\right]^{\alpha}\left[\pi\left(1-\alpha-\beta\right)+V^{*}-\frac{1}{\eta}V^{*}\right]^{\beta}$$

where

$$V^{*}=\left(1-\pi\right)\left(1-\alpha-\beta\right)-\frac{\alpha\eta}{p}>0$$

This clearly shows that the bubbleless output may be higher or lower depending on the parameters since it features more tangible capital but less intangible capital (as long as $\eta > 1$). We observe a drop in output when the bubble bursts if the second term prevails.

Rewrite:

$$\left(\left(1-\pi\right)\left(1-\alpha-\beta\right)\right)^{\alpha}\left(\pi\left(1-\alpha-\beta\right)\right)^{\beta}<\left[\frac{\alpha\eta}{p}\right]^{\alpha}\left[\pi\left(1-\alpha-\beta\right)+\left(1-\pi\right)\left(1-\alpha-\beta\right)\frac{\eta-1}{\eta}-\frac{\eta-1}{p}\alpha\right]^{\beta}$$

We now look at the limit for $\alpha \to 0$ but keeping $v \equiv \alpha + \beta$ constant (so $\beta \to v$). the left-hand side converges to:

$$\left(\pi\left(1-v\right)\right)^{v}$$

and the right-hand side converges to:

$$\left[\pi\left(1-v\right)+\left(1-\pi\right)\left(1-v\right)\frac{\eta-1}{\eta}\right]^{v}$$

Since $\frac{\eta-1}{\eta} > 0$ and $0 < v < 1$, it is clear that

$$\left(\pi\left(1-v\right)\right)^{v}<\left[\pi\left(1-v\right)+\left(1-\pi\right)\left(1-v\right)\frac{\eta-1}{\eta}\right]^{v}$$

By continuity, for α small enough, after the bubble explodes the economy converges to a steady state in which output is in fact lower. □

Log-utility of Consumption for Households

Deterministic Bubbles

Its is now proved that the main results of the paper are not materially affected by the benchmark specification where agents' savings decisions are exogenously determined. To see this, let us introduce a meaningful savings choice by assuming that households and entrepreneurs consume both during their middle age and when they are old. More specifically, suppose that the lifetime utility of an individual from generation t is given by

$$U\left(c_t\left(t+1\right), c_t\left(t+2\right)\right) = ln\left(c_t\left(t+1\right)\right) + ln\left(c_t\left(t+2\right)\right)$$

At time $t+1$, the agent's intertemporal allocation problem is given by:

$$max\left\{U\left(c_t\left(t+1\right), c_t\left(t+2\right)\right)\right\}$$

$$s.t.$$

$$c_t\left(t+1\right) + \frac{c_t\left(t+2\right)}{\tilde{R}\left(t+2\right)} \leq w\left(t+1\right)$$

where $\tilde{R}\left(t+2\right)$ is the rate of return each individual faces on her investment opportunity. Households always obtain the interest rate paid on physical capital R, while entrepreneurs enjoy a return given by IRR.

The solution to this maximization problem features:

$$c_t\left(t+1\right) = \frac{w\left(t+1\right)}{2}$$

Hence, with log utility of consumption in the last two periods of life, agents always save half of their labor income during middle age. Note that the solution is independent of the rate of return prevalent at the time savings are chosen, implying that the fraction of income saved is identical for both households and entrepreneurs. This results from a property of the log-utility specification, where the income and substitution effects associated with changes in interest rates offset each other perfectly.

In light on this finding, the dynamic equations describing the evolution of the economy shall not change substantially. Indeed, the only modification regards the equilibrium in the savings market. To see this, note that, in any bubbleless equilibrium, the savings market-clearing conditions is:

$$Z_t\left(t+1\right)+K_t\left(t+1\right)=\pi\frac{w\left(t+1\right)}{2}+\left(1-\pi\right)\frac{w\left(t+1\right)}{2}=\frac{w\left(t+1\right)}{2}$$

When financing constraints bind, the following conditions hold:

$$Z_t\left(t+1\right)=\pi\frac{w\left(t+1\right)}{2}$$

$$K_t\left(t+1\right)=\left(1-\pi\right)\frac{w\left(t+1\right)}{2}$$

Moreover, as shown in the proofs of various propositions, binding financing constraints also imply:

$$\alpha Z_t\left(t+1\right)\leq\beta K_t\left(t+1\right)$$

Therefore, it becomes clear that financial frictions will bind whenever

$$\frac{\beta}{\alpha}\geq\frac{\pi}{\left(1-\pi\right)}$$

and they will be slack if the inequality is reversed. This threshold is identical to the original threshold separating the B and NB regions of the parameter space in section 4.

The same logic can be easily applied to all other results in subsection 4.1. Moreover, it is also immediate that all results in subsection 4.2 hold as well, with small modifications in some of the expressions. For example, proposition 3 would be adjusted with a slightly different expression for the bubble-to-output ratio in the steady-state:

$$B^* = \left[\frac{(1-\pi)(1-\alpha-\beta)-2\alpha}{2}\right]$$

where the numerator is positive for α sufficiently small. Note that this is maximum bubble-to-output ratio the economy can sustain in equilibrium. It is smaller than the corresponding bubble in the original specification, since middle-age consumption reduces the supply of savings necessary to sustain any bubble.

Finally, it is important to emphasize that, in the present context, the AMSZ criterion is satisfied whenever $\alpha + \beta \geq \frac{1}{3}$. This can be easily verified from an adjustment of the proof of Lemma 3 to consider agents' savings decision. In the original specification, the AMSZ threshold was higher ($\alpha + \beta \geq \frac{1}{2}$). Hence, a larger set of economies shall satisfy the AMSZ condition when savings are endogenized. This result reinforces the importance of active financing constraints as underpinnings of asset bubbles in a context where economies are efficient.

Stochastic Bubbles

The case of stochastic bubbles is slightly more involved, since the log-utility of consumption introduces risk aversion in investors' portfolio choice problem. Despite the solution to this case being different from the closed-form expressions derived in subsection 4.1 and 4.2, the basic intuition remains the same. To illustrate the main points, we will restrict ourselves to equilibria where neither households nor entrepreneurs can short the risky bubble.

The first aspect to be noted is that, with a strictly concave utility of consumption in the last period, the optimization problem of households from generation t is given by:

$$max_{c,\theta}\{ln\left(c_t\left(t+1\right)\right)+ln\left[\left(w\left(t+1\right)-c_t\left(t+1\right)\right)\right]+pln\left[\theta_t^h\left(t+1\right)R^*\left(t+2\right)+\left(1-\theta_t^h\left(t+1\right)\right)R\left(t+2\right)\right]$$

$$+ (1-p) \, ln \left[\left(1 - \theta_t^h (t+1)\right) R(t+2) \right] \}$$

where $\theta_t^h (t+1)$ is the fraction of wealth invested in the risky bubble and $R^* (t+2)$ denotes the gross return on the risky bubble if it does not burst, whereas $R(t+2)$ is the riskless interest rate they obtain by investing in tangible capital.

The solution to this problem involves:

$$c_t (t+1) = \frac{w(t+1)}{2}$$

$$\theta_t^h (t+1) = \frac{p\tilde{R}(t+2) - R(t+2)}{\tilde{R}(t+2) - R(t+2)}$$

As argued in subsection 4.3, in any bubbly conditional steady-state the return $\tilde{R}(t+2)$ has to be equal to the growth rate of the economy $(1+g)$. A smaller $\tilde{R}(t+2)$ leads to an ever declining bubble-to-output ratio, whereas a larger rate of return leads to explosive paths in finite time. Therefore, a bubbly conditional steady-state features:

$$\theta^h = \frac{p(1+g) - R}{(1+g) - R}$$

The corresponding portfolio allocation of entrepreneurs θ^e is even simpler: in equilibrium, they will not hold any fraction of the bubble (in fact they would short it if they could). The reason is that, for an economy that satisfies the AMSZ criterion, the steady-state IRR has to be larger than the growth rate $(1+g)$, making the risky bubble a dominated investment opportunity for entrepreneurs.

Extending this logic, we can prove that there cannot exist an unconstrained conditional steady-state with a positive bubble for economies that satisfy the $AMSZ$ benchmark. To see this, consider an economy with a bubbly conditional steady-state where financing constraints are slack. This implies $R = IRR$. Hence, both households and entrepreneurs allocate an identical fraction of their wealth to invest in the risky bubble: $\theta^h = \theta^e = \theta > 0$. This requires

$p(1+g) > R = IRR$, which yields:

$$\pi IRR + (1-\pi) R < (1+g)$$

Following the same steps in the proof of Lemma 3, we can easily verify that this inequality violates the $AMSZ$ condition.

Based on these considerations, a bubbly conditional steady-state will have binding financial constraints. Consequently,

$$\alpha \hat{Z} < \beta \hat{K}$$

or

$$\frac{\beta}{\alpha} > \frac{\pi}{(1-\pi)(1-\theta^h)}$$

For any $\theta^h \in (0,1)$, this inequality will be satisfied provided β is sufficiently large relative to α. The new threshold for the ratio $\frac{\beta}{\alpha}$ is higher than in the original specification as well as in the log-utility case without uncertainty. In other words, the combination of uncertainty and risk aversion makes it harder for bubbles to be sustained in equilibrium. In the current framework, bubbly equilibria demand an even higher importance of intangibles relative to physical capital.

In steady-state, the riskless interest rate equals:

$$R = \alpha \frac{\hat{Y}}{\hat{K}}$$

or

$$R = \frac{2\alpha(1+g)}{(1-\alpha-\beta)(1-\pi)(1-\theta^h)}$$

Substituting this condition in the expression for θ^h, we obtain:

$$\theta^h = \frac{p(1-\pi)(1-\alpha-\beta)\left(1-\theta^h\right) - 2\alpha}{(1-\pi)(1-\alpha-\beta)(1-\theta^h) - 2\alpha}$$

First, note that for θ^h to be positive, we need

$$\frac{2\alpha}{(1-\pi)(1-\alpha-\beta)(1-\theta^h)} \leq p$$

a condition that also guarantees that $R \leq (1+g)$. This will always be the case provided that α is small enough.

Defining $\psi = (1-\pi)(1-\alpha-\beta)$ and noting that $p\psi \geq 2\alpha$, we conclude that θ^h satisfies the following quadratic equation:

$$\psi \left(\theta^h\right)^2 - [\psi + p\psi - 2\alpha]\,\theta^h + [p\psi - 2\alpha] = 0$$

whose solutions are:

$$\theta^h = \frac{[\psi + p\psi - 2\alpha] \pm \sqrt{[\psi + p\psi - 2\alpha]^2 - 4\psi\,[p\psi - 2\alpha]}}{2\psi}$$

This expression can be rewritten as:

$$\theta^h = \frac{[\psi + p\psi - 2\alpha] \pm \sqrt{\psi^2 - 2\psi\,[p\psi - 2\alpha] + [p\psi - 2\alpha]^2}}{2\psi}$$

whose solutions are:

$$\theta^h_+ = 1$$

and

$$\theta^h_- = \frac{p\psi - 2\alpha}{\psi}$$

We can immediately rule out the case $\theta^h = 1$. It implies that households are fully invested in the bubble, which requires entrepreneurs to invest in both physical and intangible capital in equilibrium. But in this case, financing constraints are slack and $R = IRR$. As argued before,the portfolio choices of households and entrepreneurs are identical when constraints are slack, which means entrepreneurs would fully invest in the bubble as well. Under those circumstances, both K and Z would be zero, and so would be the bubble.

Therefore, there is a unique feasible solution for households' portfolio choice in a bubbly conditional steady-state:

$$\theta^{h*} = \frac{p\psi - 2\alpha}{\psi} \in (0,1)$$

Given θ^{h*}, the expression for the interest rate in the conditional steady-state becomes:

$$R^{*} = \frac{2\psi\alpha\left(1+g\right)}{\left(1-\alpha-\beta\right)\left(1-\pi\right)\left(\psi\left(1-p\right)+2\alpha\right)}$$

The value of the bubble follows:

$$B\left(t+1\right) = \frac{\theta^{h*}\psi}{2} Y_{t}\left(t+1\right)$$

Hence, the bubble-to-output ratio in a conditional steady-state is:

$$B^{*} = \frac{p\psi - 2\alpha}{2}$$

The following set of equations determines the steady-state values of $\hat{Y}$, $\hat{K}$ and $\hat{Z}$:

$$\hat{Z}^{*} = \frac{\pi\psi}{2\left(1-\pi\right)\left(1+g\right)}\hat{Y}^{*}$$

$$\hat{K}^{*} = \frac{\left(1-\theta^{h*}\right)\psi}{2\left(1+g\right)}\hat{Y}^{*}$$

$$\hat{Y}^{*} = \left(\hat{K}^{*}\right)^{\alpha}\left(\hat{Z}^{*}\right)^{\beta}$$

Finally, the equilibrium $IRR = \beta\frac{\hat{Y}^{*}}{\hat{Z}^{*}}$ has to be larger than $(1+g)$ for entrepreneurs not to hold the bubble:

$$IRR = \frac{2\beta\left(1+g\right)}{\pi\left(1-\alpha-\beta\right)}$$

Hence, we have proved the existence of a bubbly conditional steady-state subject to the following restrictions:

1. The economy satisfies the $AMSZ$ benchmark: $\alpha + \beta \geq \frac{1}{3}$.
2. Financial frictions are binding: $\frac{\beta}{\alpha} \geq \frac{\pi}{(1-\pi)\left(1-\theta^{h*}\right)}$.
3. The bubble is positive: $p\,(1-\pi)\,(1-\alpha-\beta) \geq 2\alpha$.
4. Entrepreneurs do not want to hold the bubble: $\beta \geq \frac{\pi}{2+\pi} - \frac{\pi\alpha}{2+\pi}$:

It is easy to verify that these conditions will be satisfied provided β is large and α is small.

The complementarity between technological progress and foreign inflows of funds: an example.

In this example, we analyze the impact of foreign inflows of funds in the domestic economy. We formalize the intuition presented in Subsection 5.3 where we argued that the savings glut explanation can complement the technology-based approach as a structural condition behind asset price bubbles.

Consider a foreign economy where the representative agent of each generation lives for two periods - we assume the foreign economy is represented by an OLG model as well. Every generation is born with an endowment Y^f. For simplicity, we assume that the representative agent in the foreign economy only consumes in the last period of her life. In the first period, she has no choice but to invest her endowment abroad, earning the same rate of return paid to domestic households R – that is, the foreign individual cannot overcome domestic moral hazard problems and is, thus, prevented from investing in intangible capital**.** So in equilibrium all the endowment of the foreign country will be invested in the domestic economy's capital stock. We assume that Y^f and Y are perfect substitutes and can be exchanged one-to-one at no cost. The endowment process Y^f grows at a rate $g^f(t)$ across generations, which will be carefully calibrated in order to facilitate the demonstration of the main intuition.

Let us focus on the dynamics of the bubbleless economy. The domestic economy is assumed to satisfy the $AMSZ$ condition, and we require it to be in the B region of the parameter space. Naturally, because the inflow of foreign savings Y^f is always positive, and given the restriction that foreigners cannot overcome domestic frictions, financing constraints will bind at all times.[2] The positive inflow of funds increases physical capital in all periods, driving its marginal productivity down and increasing the attractiveness of intangibles.

Under those assumptions, the domestic economy evolves according to the following set of equations:

$$K_t(t+1) = (1-\pi)(1-\alpha-\beta) Y_{t-1}(t+1) + Y^f(t+1)$$

$$Z_t(t+1) = \pi(1-\alpha-\beta) Y_{t-1}(t+1)$$

$$Y_t(t+2) = A(t+2) K_t(t+1)^{\alpha} Z_t(t+1)^{\beta}$$

$$R(t+2) = \alpha \frac{Y_t(t+2)}{K_t(t+1)}$$

The evolution of the economy can be reduced to:

$$Y_t(t+2) = A(t+2)\left[(1-\pi)(1-\alpha-\beta) Y_{t-1}(t+1) + Y^f(t+1)\right]^{\alpha}\left[\pi(1-\alpha-\beta) Y_{t-1}(t+1)\right]^{\beta}$$

In order to obtain a steady-state in which the foreign economy does not disappear or explode in relative terms, we need to calibrate the process Y^f accordingly. For example, suppose that Y^f grows at all times at a rate $1+g$, the same growth rate of the domestic

[2] The overall net inflow of funds at date t will be negative if $R(t) < \left(1+g^f(t)\right)$, since the outflows are given by $R(t) Y^f(t-1)$. However, only the inflows $Y^f(t)$ affect capital accumulation at time t, since outflows are withdrawn from the consumption of old individuals of the previous generation.

economy in the long-run. Then, we have:

$$\hat{Y}_t\left(t+2\right) = [\left(1-\pi\right)\frac{\left(1-\alpha-\beta\right)}{1+g}\hat{Y}_{t-1}\left(t+1\right) + \hat{Y}^f\left(t+1\right)]^{\alpha}[\pi\left(1-\alpha-\beta\right)\hat{Y}_{t-1}\left(t+1\right)]^{\beta}$$

where

$$\hat{Y}^f\left(t+1\right) \equiv \frac{Y^f(t+1)}{A(t+1)^{\frac{1}{1-\alpha-\beta}}}$$

which is constant at some level $\hat{Y}^f$ determined by the initial conditions. It is then easy to see that there is a steady-state such that $\hat{Y}$ solves:

$$\hat{Y} = [\left(1-\pi\right)\frac{\left(1-\alpha-\beta\right)}{1+g}\hat{Y} + \frac{\hat{Y}^f}{1+g}]^{\alpha}[\pi\left(1-\alpha-\beta\right)\frac{\hat{Y}}{1+g}]^{\beta}$$

$$\hat{K} = \frac{\left(1-\pi\right)\left(1-\alpha-\beta\right)\hat{Y} + \hat{Y}^f}{1+g}$$

and

$$R = \alpha\frac{\hat{Y}}{\hat{K}}$$

Without loss of generality, let us define

$$\theta \equiv \frac{\hat{Y}^f}{\hat{Y}}$$

where θ is a function of $\hat{Y}$ at the steady state.

This allows us to solve the system of equations above to arrive at

$$\hat{Y} = \left\{\left[\frac{\left(1-\pi\right)\left(1-\alpha-\beta\right)+\theta}{\left(1+g\right)}\right]^{\alpha}\left[\frac{\pi\left(1-\alpha-\beta\right)}{\left(1+g\right)}\right]^{\beta}\right\}^{\frac{1}{1-\alpha-\beta}}$$

where again θ is defined implicitly as a function of $\hat{Y}$.

$$\hat{K} = \left[\frac{\left(1-\pi\right)\left(1-\alpha-\beta\right)+\theta}{\left(1+g\right)}\right]\hat{Y}$$

and

$$R = \frac{\alpha(1+g)}{(1-\pi)(1-\alpha-\beta)+\theta}$$

Based on the equations above, we can choose any positive value for θ if we calibrate $\hat{Y}^f$ appropriately. In fact, any choice of $\theta \geq 0$ results in a unique value for the domestic output in steady-state $\hat{Y}$. Then, the corresponding steady-state value of the foreign endowment $\hat{Y}^f$ can be directly recovered from the definition of θ.

As it should be clear by now, the fundamental condition for rational bubbles to be sustained in equilibrium is that, in the bubbleless steady-state, the interest rate in the domestic economy R is below its growth rate $(1+g)$. Hence, to check for the interaction between technological progress and foreign inflows of funds, and their implications for bubbles, we analyze how the steady-state interest rate varies with θ and the technological parameters. Indeed, note that $R < (1+g)$ if and only if

$$\theta > \alpha - (1-\pi)(1-\alpha-\beta)$$

This inequality will hold provided that θ is large enough. Conversely, for any positive θ - no matter how small - and for any $(\alpha+\beta) \geq 0.5$, the inequality will be satisfied provided the ratio $\frac{\beta}{\alpha}$ is large enough. Finally, it is clear that the required threshold $\frac{\beta}{\alpha}$ is decreasing in θ. Therefore, foreign inflows of funds and technological progress complement each other in driving the domestic interest rate below the growth rate of the economy in the long-run. Their combination is fuel to bubbly episodes.

Our example is admittedly stylized but it still retains some interesting properties. First it assumes that the size of the foreign economy stabilizes relative to the domestic one in the long-run, so it does not rely on one economy growing faster than the other forever. Second, the example does not impose any ad-hoc restrictions on the evolution of the current account balance.[3] Hence, it is clear that the intuition built here can be carried over to more general

[3] In the bubbleless steady-state, the domestic economy will run a permanent current account surplus if and only if $R > (1+g)$.

environments.

Calibration Details

Our model has only four parameters to calibrate: α, β, π, n.

- We calibrate the parameters of the production function, α and β, in a way that guarantees that AMSZ's criterion is satisfied. This imposes $\alpha + \beta > \frac{1}{2}$. We pick $\alpha = 0.27$, $\beta = 0.25$ as the starting point of our calibration (which lies in the NB region). Note that in our simplified model we need to have $\alpha+\beta$ high enough in order for the economy to be producing more resources than are entering the economy, since in the model the whole fraction $1-\alpha-\beta$ of output (paid as wages) reenters the economy as investment. Of course, enriching the model by allowing consumption in the middle period and removing full depreciation would allow us to achieve a better match of the levels of these parameters with those observed in the data.

- In our model, π represent the fraction of entrepreneurs in the population. In a more general interpretation, it is related to the leverage of the corporate sector: the lower π, the larger the fraction of output saved by households and lent to entrepreneurs relative to their own funds, and therefore the higher the leverage ratio $\frac{(1-\pi)}{\pi}$. Picking $\pi = 0.6$ leads to a leverage ratio of about $\frac{2}{3}$, consistent with the evidence for the United States.

- We choose $n = 0.01$ per period.